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13. ABSTRACT (Maximum 200 words)

In estimating the lifetime distribution of a product, the following periodic inspection plan is put forward. Select a random sample of n units of the product and set them to work. Choose an integer $m \geq 1$, numbers $0 < t_1 < t_2 < \ldots < t_m$, and inspect the units at times $t_1, t_2, \ldots, t_m$. The data consist of number of units that fail between every two consecutive inspection times. The choice of m and numbers $t_1, t_2, \ldots, t_m$ are discussed in this paper. Some graphical methods are presented to identify the parametric model of the underlying lifetime distribution based on the interval-censored data gathered. A new method of estimating the parameters of the lifetime distribution based on Linear Model Theory is presented and compared with the method of maximum likelihood.

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INTERVAL-CENSORED DATA

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INTERVAL-CENSORED DATA

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SUMMARY

In estimating the lifetime distribution of a product, the following periodic inspection plan is put forward. Select a random sample of n units of the product and set them to work. Choose an integer $m \geq 1$, numbers $0 < t_1 < t_2 < \ldots < t_m$, and inspect the units at times $t_1, t_2, \ldots, t_m$. The data consist of number of units that fail between every two consecutive inspection times. The choice of m and numbers $t_1, t_2, \ldots, t_m$ are discussed in this paper. Some graphical methods are presented to identify the parametric model of the underlying lifetime distribution based on the interval-censored data gathered. A new method of estimating the parameters of the lifetime distribution based on Linear Model Theory is presented and compared with the method of maximum likelihood.

Some key words: Asymptotic Variance, Delta method, Distributions: Exponential, Extreme Value, Logistic, Lognormal, Pareto and Weibull, Graphical Methods, Inspection times, Interval-censored data, Lifetime distribution, Linear Model, Loss of Information, Method of Maximum Likelihood, Relative Efficiency, Simulations, Tri-diagonal Matrix.

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1. INTRODUCTION

In Reliability studies, one of the basic problems is the estimation of the lifetime distribution of a product. In medical research, it is the estimation of the lifetime distribution of a patient under remission of a particular disease. For example, in engineering, as a part of quality control exercise, one would like to estimate the lifetime distribution of a day's production of lightbulbs from a particular process. In medical research, one of the most common problems is to compare the lifetime distributions of patients who have undergone two different treatment regimens. One of the regimens is, usually, a conventional one and the other one a new treatment. As an example, consider the following problem we have come across recently. The conventional treatment for breast cancer patients has been and continues to be the one involving chemotherapy. Chemotherapy not only destroys cancer cells, but also good cells. The National Cancer Institute started research into a new treatment: remove a substantial part of bone marrow from the patient, store it, and then apply high doses of chemotherapy until all cancerous cells are destroyed. The bone marrow is inserted back into the bones of the patient. The patient is kept in an isolated ward until the bone marrow grows back into a sufficient quantity. One of the concerns of researchers is to compare the lifetime distribution of patients who have undergone the new treatment with that of patients who have undergone the conventional treatment.

Ideally, in the context of the lightbulb example, one takes a random sample of n bulbs, sets them to work, and follows them until all of them die. The observed lifetimes are usually written as $T_1, T_2, \ldots, T_n$. The next task is to identify a parametric distribution that fits the data reasonably well. A variety of probability graph papers such as Exponential, Weibull, Lognormal, or Logistic distributions can be used to plot the empirical distribution function of the data. The empirical distribution function of the data is defined by

$$F(t) = \#\{1 \le i \le n;\ T_i \le t\}/n, \quad 0 < t < \infty.$$

If the points $(T_i, F(T_i))$, i=1, 2, ..., n lie more or less on a line in the graph, the corresponding parametric probability model is fitted to the data.

The full sampling plan of observing each and every unit in the sample until it dies is not practical in many cases. Instead, one can select m time points $0 < t_1 < t_2 < ... < t_m$ and inspect the units only at each of these m time points. In such a case, the data consist of the following: X_1 = # units that failed during $(0, t_1]$, X_i = # units that failed during $(t_{i-1}, t_i]$, i=2, 3, ..., m, and X_{m+1} = # units that failed during (t_m, ∞). There are some problems to be resolved with this periodic inspection plan. (1) How many inspections m are to be made? (2). How are the inspection times to be selected?

The first question is addressed in Section 2. The maximum likelihood estimation in multinomial models is discussed in Section 3. In the case when all the inspection intervals are of the same length t_0, we will discuss an optimal choice of t_0 for a given value of m for some parametric models of lifetime distribution in Section 4. The loss of information that results in adopting the periodic inspection plan in lieu of the full sampling plan is discussed in Section 5. The maximum likelihood estimation for interval-censored data is discussed in Section 6. In Section 7, we establish some results in matrix algebra that are needed to develop the linear model approach. Graphical analysis for the periodic inspection plan is discussed in Section 8. In Section 9, a linear model approach is developed to estimate the lifetime distribution for general periodic inspection plans. Some simulation studies are conducted for comparing the methods of maximum likelihood and linear model in small samples in Section 10. Finally, in Section 11, some useful pointers from this study are presented.

Kulldorf (1961) is one of the earliest researchers devoted analysis of interval-censored data and grouped data. Kulldorf (1961) showed theexistence and uniqueness of maximum likelihood estimates for interval-censored data when the underlying distribution is exponential. He also discussed the optimal length of equi-spaced inspection intervals based on the asymptotic variance of the maximum likelihood estimator. Nelson (1977) extended the work of kulldorf (1961) on exponential distribution. See also Nelson (1982) and Lawless (1982). Meeker (1986) disscused optimal length of equi-spaced intervals in log time for

estimating a particular quantile of a Weibull distribution. Ostrouchov and Meeker (1986) discussed finite sample properties of maximum likelihood estimators of parameters of a Weibull distributionin environment of interval-censored data.

Kraft (1992) discussed estimation of parameters of an exponential distribution when inspection times are generated by a poisson process. Interval-cesored data appear in ecological and biological field studies. Some discussion of methods is available in journals devoted to this area. See Derleth and Sepik (1990), Johnson (1979), and Johnson and Christensen (1986). For general survays, see Sampford (1952) and Deddens and Koch (1988). For analysis of interval-censored data from a nonparametric angle, see Turnbull (1976), Pierce, Stewart and Kopecky (1979), and Akritas (1988). In the frame work proportional hazards and interval-censored data see Satten (1996).

2. HOW MANY INSPECTIONS?

If one wants to terminate the study after a certain number m of inspections, one would like to determine the minimum number of inspections required in order to estimate the parameters of the underlying distribution. The answer to this question is that the number inspections must be at least equal to the number of parameters to be estimated. This statement is amplified in the following examples.

Example 1: Let T be the underlying lifetime random variable with Logistic distribution. Its probability density function is given by

$$f_T(t) = \begin{cases} \dfrac{\alpha\gamma t^{\gamma-1}}{(1+\alpha t^{\gamma})^2} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\gamma > 0$ are the unknown parameters of the model. Suppose only one inspection is made, i.e., m=1. The observable random variable is then X_1 = #

units that failed during $(0, t_1]$. Note that X_1 has a Binomial distribution with n = sample size and $p=\frac{\alpha t_1^{\gamma}}{1+\alpha t_1^{\gamma}}$. The likelihood of the data $X_1 = x$ is given by

$$L(\alpha,\gamma)=\binom{n}{x}\left[1-\frac{1}{1+\alpha t_1^{\gamma}}\right]^{x}\left[\frac{1}{1+\alpha t_1^{\gamma}}\right]^{n-x}.$$

The likelihood equations

$$\frac{\partial}{\partial\alpha}\ell n\, L(\alpha,\gamma)=0$$

and

$$\frac{\partial}{\partial\gamma}\ell n\, L(\alpha,\gamma)=0$$

lead to the same equation

$$\frac{x}{n}=\frac{\hat{\alpha} t_1^{\hat{\gamma}}}{1+\hat{\alpha} t_1^{\hat{\gamma}}}.$$

There is no unique solution.

Example 2: Let T be the underlying lifetime random variable with Weibull distribution. Its probability density function is given by

$$f_T(t) = \begin{cases} \lambda\gamma t^{\gamma-1}e^{-\lambda t^{\gamma}} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ and $\gamma > 0$ are the unknown parameters of the model. Suppose only one inspection is made, i.e., m=1. Then X_1 has a Binomial distribution with n = sample size and $p = 1-e^{-\lambda t_1^{\gamma}}$. The likelihood of the data $X_1 = x$ is given by

$$L(\lambda,\gamma)=\binom{n}{x}\left[1-e^{-\lambda t_1^{\gamma}}\right]^{x}\left[e^{-\lambda t_1^{\gamma}}\right]^{n-x}.$$

The likelihood equations

$$\frac{\partial}{\partial\lambda}\ell n\, L(\lambda,\gamma)=0$$

and

$$\frac{\partial}{\partial \gamma} \ell n\, L(\lambda, \gamma) = 0$$

lead to the same equation

$$\frac{x}{n} = 1 - e^{-\hat{\lambda} t_1^{\hat{\gamma}}}.$$

There is no unique solution.

3. MAXIMUM LIKELIHOOD ESTIMATION IN MULTINOMIAL MODELS

Every periodic inspection plan gives rise to multinomial data. The multinomial probabilities are usually functions of a finite number of parameters. It is important to identify the asymptotic distribution of the maximum likelihood estimators of these parameters for our work. The asymptotic theory for the multinomial distribution is somewhat different from the asymptotic theory of iid random variables. The following result is the bulwark of maximum likelihood estimation in multinomial distribution. The conditions of the theorem have undergone some changes over the years, but the conclusions are essentially the same. See Fisher (1928), Rao (1958, 1973), Birch (1964), and Sen and Singer (1993). We report the version given by Sen and Singer (1993. p. 253).

Suppose X_1, X_2, ..., X_m, X_{m+1} have a multinomial distribution with probabilities $p_1(\tilde{\theta})$, $p_2(\tilde{\theta})$, ..., $p_m(\tilde{\theta})$, $p_{m+1}(\tilde{\theta})$, where

$$X_1 + X_2 + ... + X_m + X_{m+1} = n,$$

and each $p_i(\tilde{\theta})$ is a function of s parameters θ_1, θ_2, ..., θ_s, with $s < m+1$. The parameter vector $\tilde{\theta} = (\theta_1, \theta_2, ..., \theta_s)$ of the distribution is assumed to belong to a known parameter space Θ, an open subset of R^s. Assume that each $p_i(\tilde{\theta})$ is positive for all $\tilde{\theta}$ and admits continuous partial derivatives up to the order 2 with respect to the components of $\tilde{\theta}$. Let $\hat{\tilde{\theta}}$ be the maximum likelihood estimator, i.e., $\hat{\tilde{\theta}} = \hat{\tilde{\theta}}(X_1, X_2, ..., X_m, X_{m+1})$ maximizes

$$L(\tilde{\theta}) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m+1} \left[p_i(\tilde{\theta})\right]^{X_i}.$$

Let

$$a_{ij}(\tilde{\theta}) = \left[p_i(\tilde{\theta})\right]^{-1/2} \frac{\partial}{\partial \theta_j} p_i(\tilde{\theta}),\ i = 1,\ 2,\ \ldots,\ m+1,\ j = 1,\ 2,\ \ldots,\ s.$$

Let $A = A(\tilde{\theta}) = (a_{ij}(\tilde{\theta}))$. Assume that A is of full rank s. Let $J(\tilde{\theta}) = nA'A$.

<u>Theorem 3.1</u> The asymptotic distribution of $(\hat{\tilde{\theta}} - \tilde{\theta})$ under $\tilde{\theta} \varepsilon \Theta$ is the multivariate Normal distribution with mean vector $\tilde{0}$ and dispersion matrix $\frac{1}{n}(A'A)^{-1}$.

Some discussion is in order on Theorem 3.1. Imitating the computation of asymptotic variance of the maximum likelihood estimator in the iid case, one can compute

$$J(\tilde{\theta}) = E_{\tilde{\theta}}(-H(\tilde{\theta})),$$

where

$$H(\tilde{\theta}) = \left(\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell n L(\tilde{\theta})\right).$$

We have observed that $J(\tilde{\theta}) = H(\tilde{\theta})$. This observation gives an alternative way of obtaining the asymptotic variance-covariance matrix of the maximum likelihood estimator for the multinomial case.

4. OPTIMAL CHOICE OF THE LENGTH OF EQUI-SPACED INSPECTION INTERVALS

In this section, we assume that the inspection intervals $(0, t_1]$, $(t_1, t_2]$, ..., $(t_{m-1}, t_m]$ are of equal length with common length equal to t_0. We determine the optimal value of t_0 based on the asymptotic variance of the maximum likelihood estimators of the parameters of the model. Assume that the underlying lifetime distribution is Exponential (θ), $\theta > 0$. The probability density function is given by

$$f_T(t;\theta) = \theta e^{-\theta t}, \quad t > 0.$$

The likelihood of the data $X_1, X_2, ..., X_m, X_{m+1}$ is given by

$$L(\theta) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[e^{-\theta[(i-1)t_0]} - e^{-\theta(it_0)} \right]^{X_i} \left[e^{-\theta m t_0} \right]^{X_{m+1}}$$

The maximum likelihood estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n = \frac{1}{t_0} \ell n \left(\frac{mn + \sum_{i=1}^{m} (i-m) X_i}{mn + \sum_{i=1}^{m} (i-1-m) X_i} \right).$$

with asymptotic variance

$$\operatorname{Var}(\hat{\theta}_n) = \frac{(e^{\theta t_0} - 1)^2}{n t_0^2 e^{\theta t_0} (1 - e^{-m\theta t_0})}.$$

The formulas presented here match with those of Nelson (1977). For a given value of θ and m, $n\operatorname{Var}(\hat{\theta}_n)$ is minimized with respect to t_0. nelson (1977) provides optimal values of θt_0. We tabulate the optimal values of t_0 for set of θ and m values (Table 4.1). The table also lists $n\operatorname{Var}(\hat{\theta}_n)$ under the column VAR. Our table is more useful in answering several questions from a practical perspective than Nelson table. If one decides how many visits m to be made and has arough idea about the value of θ, the length t_0 of the inspection intervalscan be read off the table. If one knows the value of θ to be within 20% of some specified value θ_0, one could use the optimal length t_0 associated with θ_0 in this sampling plan without increasing the asymptotic variance substantially. We illustrate this point with a few examlpes.

Table 4.1

Optimum Inspection Interval Length t_0 For Different Values of θ and m for the case of Exponential Distribution.

θ	EX =	m = 1		m = 2		m = 3		m = 4		m = 5		m = 6	
	$1/\theta$	t_0	ASV	t_0	ASV	t_0	ASV	t_0	ASV	t_0	ASV	t_0	ASV
0.1	10.00	15.94	0.0154	12.07	0.0124	9.89	0.0114	8.47	0.0110	7.45	0.0107	0.68	0.0106
0.2	5.00	7.97	0.0618	6.04	0.0495	4.95	0.0457	4.24	0.0439	3.73	0.0429	3.34	0.0423
0.3	3.33	5.31	0.1390	4.02	0.1114	3.33	0.1029	2.82	0.0988	2.49	0.0966	2.23	0.0951
0.4	2.50	3.98	0.2471	3.02	0.1981	2.47	0.1829	2.12	0.1757	1.86	0.1717	1.67	0.1691
0.5	2.00	3.18	0.3860	2.41	0.3096	1.98	0.2858	1.67	0.2746	1.49	0.2682	1.34	0.2642
0.6	1.67	2.66	0.5559	2.01	0.4458	1.65	0.4115	1.41	0.3954	1.24	0.3863	1.11	0.3805
0.7	1.43	2.28	0.7566	1.72	0.6067	1.41	0.5601	1.21	0.5382	1.07	0.5278	0.96	0.5179
0.8	1.25	1.99	0.9882	1.51	0.7925	1.24	0.7315	1.06	0.7029	0.93	0.6867	0.84	0.6764
0.9	1.11	1.77	1.2508	1.34	1.0030	1.10	0.9259	0.94	0.8896	0.83	0.8691	0.74	0.8561
1.0	1.00	1.59	1.5441	1.21	1.2382	0.99	1.1430	0.84	1.0983	0.75	1.073	0.67	1.0569
2.0	0.50	0.80	6.1766	0.60	4.9540	0.49	4.5722	0.42	4.3934	0.37	4.2920	0.33	4.2279
3.0	0.33	0.53	13.8973	0.40	11.1442	0.33	10.2873	0.28	9.8852	0.25	9.6570	0.22	9.5128
4.0	0.25	0.40	24.7065	0.30	19.8119	0.25	18.2891	0.21	17.5736	0.19	17.1692	0.17	16.9121
5.0	0.20	0.32	38.6038	0.24	30.9560	0.20	28.5768	0.17	27.4583	0.15	26.8249	0.13	26.4277
6.0	0.17	0.27	55.5980	0.20	44.5767	0.17	41.1614	0.14	39.5407	0.12	38.6381	0.11	38.0513
7.0	0.14	0.23	75.6675	0.17	60.6782	0.14	56.0104	0.12	53.8192	0.11	52.5884	0.10	51.8107
8.0	0.12	0.20	98.8258	0.15	79.2474	0.12	73.1765	0.11	70.3200	0.09	68.6900	0.08	67.6690
9.0	0.11	0.18	125.0954	0.13	100.3374	0.11	92.5856	0.09	89.0128	0.08	86.9356	0.07	85.6700
10.0	0.10	0.16	154.4153	0.12	123.8241	0.10	114.7306	0.08	109.9294	0.07	107.3936	0.07	105.7362

We now take up the case of the Weibull distribution. The probability density function is given by

$$f_T(t) = \begin{cases} \lambda\gamma t^{\gamma-1} e^{-\lambda t^{\gamma}} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Likelihood of the data:

$$L(\tilde{\theta}) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[e^{-\lambda(i-1)t_0^{\gamma}} - e^{-\lambda i t_0^{\gamma}} \right]^{X_i} \left[e^{-\lambda m t_0^{\gamma}} \right]^{(n - \sum_{i=1}^{m} X_i)}.$$

Loglikelihood of the data:

$$\ell(\tilde{\theta}) = \text{const.} + \sum_{i=1}^{m} X_i \, \ell n \left[e^{-\lambda(i-1)t_0^{\gamma}} - e^{-\lambda i t_0^{\gamma}} \right] - \lambda m t_0^{\gamma} X_{m+1}.$$

First-order derivatives of the loglikelihood:

$$\frac{\partial \ell(\tilde{\theta})}{\partial \lambda} = \sum_{i=1}^{m} \frac{i t_0^{\gamma} e^{-\lambda(i-1)t_0^{\gamma}} - (i-1) t_0^{\gamma} e^{-\lambda i t_0^{\gamma}}}{e^{-\lambda(i-1)t_0^{\gamma}} - e^{-\lambda i t_0^{\gamma}}} X_i - m t_0^{\gamma} X_{m+1},$$

and

$$\frac{\partial \ell(\tilde{\theta})}{\partial \gamma} = \sum_{i=1}^{m} \frac{i t_0^{\gamma} (\ell n \, i t_0) e^{-\lambda(i-1)t_0^{\gamma}} - (i-1) t_0^{\gamma} (\ell n \, (i-1) t_0) e^{-\lambda i t_0^{\gamma}}}{e^{-\lambda(i-1)t_0^{\gamma}} - e^{-\lambda i t_0^{\gamma}}} X_i - \lambda m t_0^{\gamma} (\ell n \, t_0) X_{m+1},$$

These are two non-linear equations in λ and γ which can be solved by using an iterative procedure such as Newton-Raphson method. It can be shown that the information matrix is given by

$$I(\tilde{\theta}) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \lambda^2} = n\sum_{i=1}^{m} \frac{\left[(it_0)^\gamma - [(i-1)t_0]^\gamma\right]^2}{e^{-\lambda(it_0)^\gamma} - e^{-\lambda[(i-1)t_0]^\gamma}},$$

$$S_{22} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \gamma^2} = n\lambda^2\sum_{i=1}^{m} \frac{\left[(it_0)^\gamma \ell n\ it_0 - [(i-1)t_0]^\gamma \ell n\ [(i-1)t_0]\right]^2}{\left[e^{\lambda(it_0)^\gamma} - e^{\lambda[(i-1)t_0]}\right]},$$

and

$$S_{12} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \lambda \partial \gamma} = n\lambda\sum_{i=1}^{m} \frac{\left[(it_0)^\gamma \ell n\ (it_0) - [(i-1)t_0]^\gamma \ell n[(i-1)t_0]\right]\left[(it_0)^\gamma - [(i-1)t_0]^\gamma\right]}{\left[e^{\lambda(it_0)^\gamma} - e^{\lambda[(i-1)t_0]^\gamma}\right]}$$

$$= S_{21}.$$

The inverse of the information matrix is given by

$$I^{-1}(\tilde{\theta}) = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}.$$

For given values of m, λ, and γ, the trace of the information matrix, apart from n, is minimized with respect to t_0. The optimum values are tabulated in Table 4.2.

Table 4.2

Optimum Inspection Interval Length t_0 for different combination of λ, γ and m.

γ	λ	E(T)	m=2		m=3		m=4		m=5		m=6	
			t_0	ASV	t_0	ASV	t_0	ASV	t_0	ASV	t_0	ASV
0.333	1.817	1	0.69	6.0802	0.57	5.2606	0.50	4.9170	0.45	4.7150	0.42	4.5778
0.500	1.414	1	0.68	4.5459	0.56	3.6429	049	3.2957	0.44	3.1030	0.40	2.9774
1.000	1.000	1	0.73	4.6483	0.61	3.2414	0.51	2.7605	0.47	2.5122	0.42	2.3621
1.500	0.859	1	0.81	5.7329	0.66	3.8877	0.56	3.2906	0.49	3.000	0.43	2.8296
2.000	0.785	1	0.83	7.0076	0.65	4.9716	0.53	4.3037	0.45	3.9781	0.39	3.7889
0.333	1.063	5	0.71	2.9487	0.58	2.4009	0.54	2.1927	0.45	2.0710	0.44	1.9947
0.500	0.632	5	0.75	2.7628	0.65	1.9448	0.65	1.6470	0.55	1.4950	0.55	1.3956
1.000	0.200	5	3.70	3.6759	3.12	2.2243	2.73	1.7427	2.44	1.5019		
1.500	0.077	5	4.25	4.6746	3.46	2.9029	2.92	2.3457	2.53	2.0787	2.15	1.9263
2.000	0.031	5	4.21	5.9947	3.3	4.0589	2.71	3.4377	2.3	3.1379	2.00	2.9645
0.333	0.843	10	0.71	2.4615	0.61	1.9320	0.52	1.7333	0.47	1.6237	0.43	1.5523
0.500	0.447	10	1.37	2.6757	1.19	1.8026	1.08	1.4917	1.00	1.3262	0.94	1.2226
1.000	0.100	10	7.74	3.4273	6.53	2.0478	5.68	1.5928	5.06	1.3663	4.56	1.2304
1.500	0.027	10	8.53	4.5951	6.93	2.8490	5.84	2.3003	5.07	2.0376	4.48	1.8853
2.000	0.008	10	8.43	5.9753	6.6	4.0450	5.41	3.4257	4.60	3.1268	4.00	2.9539

5. LOSS OF INFORMATION

One of the questions that arises in the case of periodic inspection plan is that how much information we lose if we adopt this plan in lieu of the full sampling plan. We proceed to answer this question in the following way.

1. Evaluate the asymptotic variances of the maximum likelihood estimators of the parameters of the underlying model under each of the full sampling and periodic inspection plans.
2. Compare the variances of the plans.

A comparison is made by computing the efficiency of the full sampling relative to the periodic inspection plan. We consider only the periodic inspection plan in which the inspection intervals are equi-spaced. The relative efficiency (RE) is defined as the ratio of the variance of the estimator under the full sampling plan to that under the periodic inspection plan in the single parameter case under the optimal choice of t_0. It is defined as the ratio of the trace of the asymptotic variance-covariance matrix under the full sampling plan to that under the periodic plan again under the optimal choice of t_0 in the multi-parameter case.

First, we take up the case of the Exponential distribution with parameter $\theta > 0$. Let m stand for the number inspections and t_0 the corresponding optimal common length of the inspection intervals. For each value of m, we will compute the efficiency of the full sampling plan relative to the periodic inspection plan. Before this, we need to determine the maximum likelihood estimator of θ under the full inspection sampling plan. Under the full sampling plan, let $T_1, T_2, \ldots, T_n$ be the lifetimes of the units. The maximum likelihood estimator $\hat{\theta}_n$ of θ is given by

$$\hat{\theta}_n = \frac{n}{\sum_{i=1}^{n} T_i},$$

and the asymptotic variance of $\hat{\theta}_n$ is given by $\frac{\theta^2}{n}$.

Therefore, the efficiency of the full sampling plan relative to the periodic inspection plan is given by

$$RE = \frac{\theta^2 t_0^2 \, e^{\theta t_0} (1 - e^{-m\theta t_0})}{(e^{\theta t_0} - 1)^2}.$$

Table 5.1 shows the optimum value of t_0 and the corresponding relative efficiency for $\theta = 0.1, 0.2, \ldots, 0.9, 1, 2, \ldots, 10$ and for $m = 1, 2, 3, 4, 5, 6$.

Table 5.1

Optimum Inspection Interval Length t_0 and Relative Efficiency (RE) of Full Sampling Plan Versus Periodic Inspection Plan for Different Values of θ and m.

θ	EX =	m = 1		m = 2		m = 3		m = 4		m = 5		m = 6	
	$1/\theta$	t_0	RE	t_0	RE	t_0	RE	t_0	RE	t_0	RE	t_0	RE
0.1	10.00	15.94	0.6476	12.07	0.8076	9.89	0.8749	8.47	0.9105	7.45	0.9320	0.68	0.9461
0.2	5.00	7.97	0.6476	6.04	0.8076	4.95	0.8749	4.24	0.9105	3.73	0.9320	3.34	0.9461
0.3	3.33	5.31	0.6476	4.02	0.8076	3.33	0.8749	2.82	0.9105	2.49	0.9320	2.23	0.9461
0.4	2.50	3.98	0.6476	3.02	0.8076	2.47	0.8749	2.12	0.9105	1.86	0.9320	1.67	0.9461
0.5	2.00	3.18	0.6476	2.41	0.8076	1.98	0.8749	1.67	0.9105	1.49	0.9320	1.34	0.9461
0.6	1.67	2.66	0.6476	2.01	0.8076	1.65	0.8749	1.41	0.9105	1.24	0.9320	1.11	0.9461
0.7	1.43	2.28	0.6476	1.72	0.8076	1.41	0.8749	1.21	0.9105	1.07	0.9320	0.96	0.9461
0.8	1.25	1.99	0.6476	1.51	0.8076	1.24	0.8749	1.06	0.9105	0.93	0.9320	0.84	0.9461
0.9	1.11	1.77	0.6476	1.34	0.8076	1.10	0.8749	0.94	0.9105	0.83	0.9320	0.74	0.9461
1.0	1.00	1.59	0.6476	1.21	0.8076	0.99	0.8749	0.84	0.9105	0.75	0.9320	0.67	0.9461
2.0	0.50	0.80	0.6476	0.60	0.8076	0.49	0.8749	0.42	0.9105	0.37	0.9320	0.33	0.9461
3.0	0.33	0.53	0.6476	0.40	0.8076	0.33	0.8749	0.28	0.9105	0.25	0.9320	0.22	0.9461
4.0	0.25	0.40	0.6476	0.30	0.8076	0.25	0.8748	0.21	0.9105	0.19	0.9320	0.17	0.9461
5.0	0.20	0.32	0.6476	0.24	0.8076	0.20	0.8749	0.17	0.9105	0.15	0.9320	0.13	0.9460
6.0	0.17	0.27	0.6475	0.20	0.8076	0.17	0.8746	0.14	0.9105	0.12	0.9317	0.11	0.9461
7.0	0.14	0.23	0.6476	0.17	0.8075	0.14	0.8748	0.12	0.9105	0.11	0.9318	0.10	0.9458
8.0	0.12	0.20	0.6476	0.15	0.8076	0.12	0.8749	0.11	0.9101	0.09	0.9317	0.08	0.9458
9.0	0.11	0.18	0.6475	0.13	0.8073	0.11	0.8749	0.09	0.9100	0.08	0.9317	0.07	0.9455
10.0	0.10	0.16	0.6476	0.12	0.8076	0.10	0.8748	0.08	0.9097	0.07	0.9312	0.07	0.9458

From Table 5.1, we notice that the relative efficiency increases as m, the number of inspections, increases. Also, the relative efficiency increased by 16% if two inspections are made rather than one inspection. This increment becomes smaller as m increases. We notice also that optimum value of t_0 is approximately equal to the expected value of the lifetime distribution T when three inspections are made. So, if the lifetime distribution is Exponential, one can make three inspections at times t_0, $2t_0$, and $3t_0$, where t_0 is approximately the expected value of the lifetime distribution.

We now take up the case of the Weibull distribution. Consider an inspection plan consisting of m inspection times $t_1 < t_2 < \dots < t_m$. Let

$$X_i = \#\text{ units that failed during the time interval } [t_{i-1}, t_i),\ i=1, 2, \dots, m,$$

and

$$X_{m+1} = \#\text{ units that failed during the time interval } (t_m, \infty),$$

with the convention that $(t_{i-1}, t_i] = (0, t_i]$, for $i = 1$. The parameter vector is $\tilde{\theta} = (\lambda, \gamma)$, and the parameter space Θ is identified as $R^+ \times R^+ = \left\{\tilde{\theta} = (\lambda, \gamma);\ \lambda > 0,\ \gamma > 0\right\}$. It is clear that $X_1, X_2, \dots, X_m, X_{m+1}$ have a multinomial distribution with probabilities

$$p_i(\tilde{\theta}) = e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}},\ i = 1, 2, \dots, m,$$

and

$$p_{m+1}(\tilde{\theta}) = e^{-\lambda t_m^{\gamma}}.$$

All the conditions of Theorem 4.3.1 are met in this case. The likelihood-based information matrix $I(\theta)$ is computed after the following derivation, with the convention that $0\ell n0 = 0$.

Likelihood of the data:

$$L(\tilde{\theta}) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}} \right]^{X_i} \left[e^{-\lambda t_m^{\gamma}} \right]^{(n - \sum_{i=1}^{m} X_i)}.$$

Loglikelihood of the data:

$$\ell(\tilde{\theta}) = \text{const.} + \sum_{i=1}^{m} X_i \,\ell n \left[e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}} \right] - \lambda t_m^{\gamma} X_{m+1}.$$

First-order derivatives of the loglikelihood:

$$\frac{\partial \ell(\tilde{\theta})}{\partial \lambda} = \sum_{i=1}^{m} \frac{t_i^{\gamma} e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma} e^{-\lambda t_{i-1}^{\gamma}}}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}} X_i - t_m^{\gamma} X_{m+1},$$

and

$$\frac{\partial \ell(\tilde{\theta})}{\partial \gamma} = \lambda \sum_{i=1}^{m} \frac{t_i^{\gamma} (\ell n\, t_i) e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma} (\ell n\, t_{i-1}) e^{-\lambda t_{i-1}^{\gamma}}}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}} X_i - \lambda t_m^{\gamma} (\ell n\, t_m) X_{m+1}.$$

These are two non-linear equations in λ and γ which can be solved by using an iterative procedure such as Newton-Raphson method. To obtain the Fisher-Rao-Birch information matrix $J(\tilde{\theta})$ in this two-parameter case, we need to identify the entries,

$$a_{ij} = \begin{cases} \dfrac{1}{\sqrt{p_i(\tilde{\theta})}} \dfrac{\partial p_i(\tilde{\theta})}{\partial \lambda} & \text{if } j = 1, \\[2ex] \dfrac{1}{\sqrt{p_i(\tilde{\theta})}} \dfrac{\partial p_i(\tilde{\theta})}{\partial \gamma} & \text{if } j = 2, \end{cases}$$

$i = 1, 2, \ldots, m, m+1$.

Therefore, the matrix A is given by

$$A = A(\tilde{\theta}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{m+11} & a_{m+12} \end{bmatrix},$$

$$A'A = \begin{bmatrix} \sum_{i=1}^{m+1} a_{i1}^2 & \sum_{i=1}^{m+1} a_{i1}a_{i2} \\ \sum_{i=1}^{m+1} a_{i1}a_{i2} & \sum_{i=1}^{m+1} a_{i2}^2 \end{bmatrix},$$

$$J(\tilde{\theta}) = nA'A = \begin{bmatrix} n\sum_{i=1}^{m+1} a_{i1}^2 & n\sum_{i=1}^{m+1} a_{i1}a_{i2} \\ n\sum_{i=1}^{m+1} a_{i1}a_{i2} & n\sum_{i=1}^{m+1} a_{i2}^2 \end{bmatrix} = \begin{bmatrix} J_{11}(\tilde{\theta}) & J_{12}(\tilde{\theta}) \\ J_{21}(\tilde{\theta}) & J_{22}(\tilde{\theta}) \end{bmatrix},$$

where

$$\begin{aligned} J_{11}(\tilde{\theta}) &= n\sum_{i=1}^{m+1} a_{i1}^2 = \sum_{i=1}^{m} \frac{1}{p_i(\tilde{\theta})}\left(\frac{\partial p_i(\tilde{\theta})}{\partial \lambda}\right)^2 \\ &= n\sum_{i=1}^{m+1} \frac{1}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}}\left[-t_{i-1}^{\gamma}e^{-\lambda t_{i-1}^{\gamma}} + t_i^{\gamma}e^{-\lambda t_i^{\gamma}}\right]^2 \\ &= n\sum_{i=1}^{m} \frac{\left[t_i^{\gamma}e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma}e^{-\lambda t_{i-1}^{\gamma}}\right]^2}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}} + nt_m^{2\gamma}e^{-\lambda t_m^{\gamma}}, \end{aligned}$$

$$\begin{aligned} J_{22}(\tilde{\theta}) &= n\sum_{i=1}^{m+1} a_{i2}^2 = \sum_{i=1}^{m} \frac{1}{p_i(\tilde{\theta})}\left(\frac{\partial p_i(\tilde{\theta})}{\partial \gamma}\right)^2 \\ &= n\sum_{i=1}^{m+1} \frac{1}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}}\left[-\lambda t_{i-1}^{\gamma}(\ell n t_{i-1})e^{-\lambda t_{i-1}^{\gamma}} + \lambda t_i^{\gamma}(\ell n t_i)e^{-\lambda t_i^{\gamma}}\right]^2 \\ &= n\lambda^2\sum_{i=1}^{m} \frac{\left[t_i^{\gamma}(\ell n t_i)e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma}(\ell n t_{i-1})e^{-\lambda t_{i-1}^{\gamma}}\right]^2}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}} + n\lambda^2 t_m^{2\gamma}(\ell n t_m)e^{-\lambda t_m^{\gamma}}, \end{aligned}$$

and

$$J_{12}(\tilde{\theta}) = n\sum_{i=1}^{m+1} a_{i1}a_{i2} = \sum_{i=1}^{m}\frac{1}{p_i(\tilde{\theta})}\left(\frac{\partial p_i(\tilde{\theta})}{\partial \lambda}\right)\left(\frac{\partial p_i(\tilde{\theta})}{\partial \gamma}\right)$$

$$= n\sum_{i=1}^{m+1}\frac{\left[-t_{i-1}^{\gamma}e^{-\lambda t_{i-1}^{\gamma}} + t_i^{\gamma}e^{-\lambda t_i^{\gamma}}\right]\left[-\lambda t_{i-1}^{\gamma}(\ell n t_{i-1})e^{-\lambda t_{i-1}^{\gamma}} + \lambda t_i^{\gamma}(\ell n t_i)e^{-\lambda t_i^{\gamma}}\right]}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}}$$

$$= n\lambda\sum_{i=1}^{m}\frac{\left[t_i^{\gamma}e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma}e^{-\lambda t_{i-1}^{\gamma}}\right]\left[t_i^{\gamma}(\ell n t_i)e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma}(\ell n t_{i-1})e^{-\lambda t_{i-1}^{\gamma}}\right]}{e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}}$$

$$+ n\lambda t_m^{2\gamma}(\ell n t_m)e^{-\lambda t_m^{\gamma}}.$$

A special case of this result is of interest. Suppose all inspection intervals have the same length t_0, i.e., $t_i = it_0$, $i = 1, 2, ..., m$. It can be shown that the information matrix is given by

$$I(\tilde{\theta}) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \lambda^2} = n\sum_{i=1}^{m}\frac{\left[(it_0)^{\gamma} - [(i-1)t_0]^{\gamma}\right]^2}{e^{-\lambda(it_0)^{\gamma}} - e^{-\lambda[(i-1)t_0]^{\gamma}}},$$

$$S_{22} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \gamma^2} = n\lambda^2\sum_{i=1}^{m}\frac{\left[(it_0)^{\gamma}\ell n\, it_0 - [(i-1)t_0]^{\gamma}\ell n\,[(i-1)t_0]\right]^2}{\left[e^{\lambda(it_0)^{\gamma}} - e^{\lambda[(i-1)t_0]}\right]},$$

and

$$S_{12} = -E\frac{\partial^2 \ell(\tilde{\theta})}{\partial \lambda \partial \gamma} = n\lambda\sum_{i=1}^{m}\frac{\left[(it_0)^{\gamma}\ell n\,(it_0) - [(i-1)t_0]^{\gamma}\ell n[(i-1)t_0]\right]\left[(it_0)^{\gamma} - [(i-1)t_0]^{\gamma}\right]}{\left[e^{\lambda(it_0)^{\gamma}} - e^{\lambda[(i-1)t_0]^{\gamma}}\right]}$$

$$= S_{21}.$$

The inverse of the information matrix is given by

$$I^{-1}(\tilde{\theta}) = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix}.$$

We now obtain the information matrix of the maximum likelihood estimators of the parameters of the Weibull distribution under the full sampling plan. Let the lifetime T have a Weibull distribution. The probability density of T is given by

$$f_T(t) = \begin{cases} \lambda\gamma t^{\gamma-1} e^{-\lambda t^{\gamma}} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ and $\gamma > 0$ are the unknown parameters of the model.

Under the full sampling plan, select n units at random. Switch them on. Watch them until all of them fail. Let $T_1, T_2, \ldots, T_n$ be the lifetimes of the units 1, 2, ..., n and $t_1, t_2, \ldots, t_n$ the observed failure times. The likelihood function is given by

$$L(\lambda,\gamma) = (\lambda\gamma)^n \left[\prod_{i=1}^{n} t_i\right]^{\gamma-1} e^{-\lambda\sum_{i=1}^{n} t_i^{\gamma}}.$$

$$\ell(\lambda,\gamma) = \ell n L(\lambda,\gamma)$$

$$= n\ell n\lambda + n\ell n\gamma + (\gamma-1)\sum_{i=1}^{n}\ell n t_i - \lambda\sum_{i=1}^{n} t_i^{\gamma}.$$

Differentiate the loglikelihood with respect to λ and γ, and equate the derivatives to zero:

$$\frac{\partial\ell(\lambda,\gamma)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i^{\gamma} = 0,$$

$$\frac{\partial\ell(\lambda,\gamma)}{\partial\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}\ell n t_i - \lambda\sum_{i=1}^{n} t_i^{\gamma}\ell n t_i = 0.$$

These are two non-linear equations in λ and γ which can be solved by using an iterative procedure, such as Newton-Raphson algorithm.

The second-order partial derivatives of the loglikelihood are given by

$$\frac{\partial^2\ell(\lambda,\gamma)}{\partial\lambda^2} = -\frac{n}{\lambda^2},$$

$$\frac{\partial^2\ell(\lambda,\gamma)}{\partial\gamma^2} = -\frac{n}{\gamma^2} - \lambda\sum_{i=1}^{n} t_i^{\gamma}(\ell n t_i)^2,$$

and

$$\frac{\partial^2 \ell(\lambda,\gamma)}{\partial\lambda\partial\gamma} = -\sum_{i=1}^{n} t_i^{\gamma} \ell n t_i.$$

The negative of the expectation of the second-order partial derivatives are needed to be evaluated. The simplest one is given by

$$-E\frac{\partial^2 \ell(\lambda,\gamma)}{\partial\lambda^2} = \frac{n}{\lambda^2}.$$

The other expectations require additional work. First, we need to introduce the Digamma function (Abramowitz and Stegun, 1970).

For any $z > 0$, the Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt,$$

the Digamma function by

$$\begin{aligned}\Psi(z) &= \frac{d}{dz} \ell n \Gamma(z) \\ &= \frac{d}{dz} \ell n \int_0^{\infty} t^{z-1} e^{-t} dt \\ &= \frac{\int_0^{\infty} (\ell n t) t^{z-1} e^{-t} dt}{\Gamma(z)} \\ &= \int_0^{\infty} (\ell n t) \frac{t^{z-1} e^{-t}}{\Gamma(z)} dt,\end{aligned}$$

and the derivative of the Digamma function by

$$\begin{aligned}\Psi'(z) = \frac{d}{dz}\Psi(z) &= \int_0^{\infty} (\ell n t)\, e^{-t} dt. \frac{\Gamma(z).t^{z-1}(\ell n\, t) - t^{z-1}\Gamma'(z)}{(\Gamma(z))^2} \\ &= \int_0^{\infty} (\ell n t)^2 \frac{t^{z-1} e^{-t}}{\Gamma(z)} dt - \Psi(z) \int_0^{\infty} \frac{t^{z-1} e^{-t}}{\Gamma(z)} (\ell n t)\, dt \\ &= \int_0^{\infty} (\ell n t)^2 \frac{t^{z-1} e^{-t}}{\Gamma(z)} dt - \Psi^2(z).\end{aligned}$$

We now compute

$$-E\left(\frac{\partial^2 \ell(\lambda,\gamma)}{\partial \gamma^2}\right) = \frac{n}{\gamma^2} + n\lambda E(T^\gamma (\ell n T)^2) = \frac{n}{\gamma^2} + n\lambda \int_0^\infty (\ell n t)^2 t^\gamma \lambda\gamma t^{\gamma-1} e^{-\lambda t^{\gamma-1}} dt.$$

Let $x = \lambda t^\gamma$. Then $dx = \lambda\gamma t^{\gamma-1} dt$, and

$$E(T^\gamma (\ell n T)^2) = \int_0^\infty \left[\ell n\left(\frac{x}{\lambda}\right)^{\frac{1}{\gamma}}\right]^2 \frac{x}{\lambda}\, e^{-x} dx$$

$$= \int_0^\infty \frac{1}{\gamma^2} (\ell n\, x - \ell n\, \lambda)^2 \frac{x}{\lambda}\, e^{-x} dx$$

$$= \frac{1}{\lambda\gamma^2}\left\{\int_0^\infty (\ell n\, x)^2 x\, e^{-x} dx - 2\ell n\, \lambda \int_0^\infty (\ell n\, x)\, x\, e^{-x} dx + (\ell n\, \lambda)^2 \int_0^\infty x\, e^{-x} dx\right\}$$

$$= \frac{1}{\lambda\gamma^2}\left\{\Psi'(2) + \Psi^2(2) - 2\Psi(2)\ell n\, \lambda + (\ell n\, \lambda)^2\right\}.$$

Therefore,

$$-E\left(\frac{\partial^2 \ell(\lambda,\gamma)}{\partial \gamma^2}\right) = \frac{n}{\gamma^2}\left\{1 + \Psi'(2) + \Psi^2(2) - 2\Psi(2)\ell n\, \lambda + (\ell n\, \lambda)^2\right\}.$$

Further,

$$-E\frac{\partial^2 \ell(\lambda,\gamma)}{\partial\lambda\partial\gamma} = nE(T^\gamma(\ell n T)) = n\int_0^\infty t^\gamma (\ell n\, t)\lambda\gamma t^{\gamma-1} e^{-\lambda t^\gamma} dt.$$

Let $x = \lambda t^\gamma$. Then, $dx = \lambda\gamma t^{\gamma-1} dt$, and

$$-E\left(\frac{\partial^2 \ell(\lambda,\gamma)}{\partial\lambda\partial\gamma}\right) = n\int_0^\infty \ell n\left(\frac{x}{\lambda}\right)^{1/\gamma}\left(\frac{x}{\lambda}\right) e^{-x} dx$$

$$= \frac{n}{\lambda\gamma}\int_0^\infty (\ell n\, x - \ell n\, \lambda)\, x e^{-x} dx$$

$$= \frac{n}{\lambda\gamma}\left\{\int_0^\infty (\ell n\, x)\, x e^{-x} dx - \int_0^\infty (\ell n\, \lambda)\, x e^{-x} dx\right\}$$

$$= \frac{n}{\lambda\gamma}\{\Psi(2) - \ell n \lambda\},$$

Therefore, the information matrix is given by

$$I = n\begin{bmatrix} \dfrac{1}{\lambda^2} & \dfrac{1}{\lambda\gamma}(\Psi(2)-\ell n\lambda) \\ \dfrac{1}{\lambda\gamma}(\Psi(2)-\ell n\lambda) & \dfrac{1}{\gamma^2}\left(1+\Psi'(2)+(\Psi(2)-\ell n\lambda)^2\right) \end{bmatrix}.$$

The determinant Δ of the information matrix is given by

$$\Delta = \frac{n}{\lambda^2\gamma^2}\left(1+\Psi'(2)+(\Psi(2)-\ell n\lambda)^2\right)-\frac{1}{\lambda^2\gamma^2}(\Psi(2)-\ell n\lambda)^2$$
$$= \frac{n}{\lambda^2\gamma^2}(1+\Psi'(2)) = \frac{n\Psi'(1)}{\lambda^2\gamma^2}.$$

The inverse of the information matrix is given by

$$I^{-1} = \frac{1}{n}\begin{bmatrix} \lambda^2\left(1+\dfrac{(\Psi(2)-\ell n\lambda)^2}{\Psi'(1)}\right) & \lambda\gamma\dfrac{(\Psi(2)-\ell n\lambda)}{\Psi'(1)} \\ \lambda\gamma\dfrac{(\Psi(2)-\ell n\lambda)}{\Psi'(1)} & \dfrac{\gamma^2}{\Psi'(1)} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \text{ say.}$$

The efficiency of the full sampling plan relative to the periodic inspection plan is defined by

$$RE = \frac{\Sigma_{11}+\Sigma_{22}}{S'_{11}+S'_{22}}.$$

Table 5.2 shows the optimum value of t_0 and the corresponding relative efficiency for different combinations of γ, λ and m.

Table 5.2

Optimum Inspection Interval Length t_0 and Relative (RE) of Full Samling Plan Versus Periodic Inspection Plan for different combination of λ, γ and m.

γ	λ	E(T)	m=2		m=3		m=4		m=5		m=6	
			t_0	RE	t_0	RE	t_0	RE	t_0	RE	t_0	RE
0.333	1.817	1	0.69	0.5642	0.57	0.6521	0.50	0.6977	0.45	0.7276	0.42	0.7494
0.500	1.414	1	0.68	0.4749	0.56	0.5927	049	0.6551	0.44	0.6958	0.40	0.7252
1.000	1.000	1	0.73	0.3693	0.61	0.5296	0.51	0.6218	0.47	0.6833	0.42	0.7267
1.500	0.859	1	0.81	0.3929	0.66	0.5794	0.56	0.6846	0.49	0.7509	0.43	0.7961
2.000	0.785	1	0.83	0.4587	0.65	0.6465	0.53	0.7468	0.45	0.8483	0.39	3.7889
0.333	1.063	5	0.71	0.4364	0.58	0.5393	0.54	0.5868	0.45	0.6213	0.44	0/6451
0.500	0.632	5	0.75		0.65	0.3804	0.65	0.4497	0.55	1.4950	0.55	0.5307
1.000	0.200	5	3.70	0.2036	3.12	0.3365	2.73	0.4294	2.44	0.4983		
1.500	0.077	5	4.25	0.3007	3.46	0.4842	2.92	0.5993	2.53	0.6763	2.15	0.7300
2.000	0.031	5	4.21	0.4073	3.3	0.6016	2.71	0.7103	2.3	0.7781	2.00	0.8237
0.333	0.843	10	0.71	0.3782	0.61	0.4819	0.52	0.5371	0.47	0.5734	0.43	0.5998
0.500	0.447	10	1.37	0.2000	1.19	0.2969	1.08	0.3588	1.00	0.4034	0.94	0.4377
1.000	0.100	10	7.74	0.1935	6.53	0.3238	5.68	0.4163	5.06	0.4853	4.56	0.5389
1.500	0.027	10	8.53	0.2994	6.93	0.4829	5.84	0.5981	5.07	0.6752	4.48	0.7898
2.000	0.008	10	8.43	0.4072	6.6	0.6014	5.41	0.7102	4.60	0.7780	4.00	0.8236

Comments:

Relative efficiency increases with increasing values of m, the number of inspections. This is to be expected. The relative efficiency does not exceed 0.86 in all cases considered. If Var(T) is large, relative efficiency is poor. The relative efficiency is very much stable for m = 4, 5, and 6. Four inspections seems to be ideal. The optimum length t_0 decreases with increasing values of m. This phenomenon is natural to expect.

6. MAXIMUM LIKELIHOOD ESTIMATION FOR INTERVAL-CENSORED DATA

In this section, we discuss the method of maximum likelihood that is normally used to estimate the lifetime distribution parameters for interval-censored data. In most cases, the likelihood equations do not have explicit solutions. We record the likelihood equation and asymptotic variances of estimators for each of the six parametric distributions: Exponential, Weibull, Lognormal, Logistic, pareto, and the Extreme value. If the sample size is reasonably large, one could use the method of maximum likelihood. The equations are and formulas are in place.

In Section 8, we introduce an alternative method, the Linear Model Approach for interval-censored data, in which an explicit formula is obtainable in the estimation of the lifetime distribution parameters. We will show that the new method give consistent estimators. Recall that the inspection plan consists of m inspection times $t_1 < t_2 < \ldots < t_m$, X_i = # units that failed during the time interval $(t_{i-1}, t_i]$, i=1, 2, ..., m, and X_{m+1} = # units that failed during the time interval (t_m, ∞).

6.1 Maximum Likelihood Estimation for the Exponential Distribution

The maximum likelihood estimator $\hat{\theta}_n$ of θ, where θ is the parameter of the underlying Exponential distribution, is a solution of the likelihood equation,

$$\frac{d\ell(\theta)}{d\theta} = \sum_{i=1}^{m} X_i \frac{\left[-t_{i-1}e^{-\hat{\theta}t_{i-1}} + t_i e^{-\hat{\theta}t_i}\right]}{\left[e^{-\hat{\theta}t_{i-1}} - e^{-\hat{\theta}t_i}\right]} - (n - \sum_{i=1}^{m} X_i)t_m = 0. \qquad (6.1.1)$$

The equation is non-linear, which can be solved using an iterative procedure. The asymptotic variance of $\hat{\theta}_n$, Using Theorem 3.1, is given by

$$\mathrm{ASYVAR}(\hat{\theta}_n) = \frac{1}{n}\frac{1}{\sum_{i=1}^{m}\frac{\left[t_i e^{-\theta t_{i-1}} - t_{i-1}e^{-\theta t_i}\right]^2}{\left[e^{-\theta t_{i-1}} - e^{-\theta t_i}\right]} + nt_m^2 e^{-\theta t_m}}. \tag{6.1.2}$$

In the special case where $t_i = it_0$, (6.1.1) has an explicit solution given by

$$\hat{\theta} = \frac{1}{t_0}\ell n\left(\frac{mn + \sum_{i=1}^{m}(i-m)X_i}{mn + \sum_{i=1}^{m}(i-1-m)X_i}\right),$$

and the asymptotic variance of $\hat{\theta}_n$ in this case is given by

$$\mathrm{ASYVAR}(\hat{\theta}_n) = \frac{(e^{\theta t_0} - 1)^2}{nt_0^2 e^{\theta t_0}(1 - e^{-m\theta t_0})}.$$

6.2 Maximum Likelihood Estimation for the Weibull Distribution

The maximum likelihood estimators $\hat{\lambda}$ and $\hat{\gamma}$ of λ and γ, respectively, where λ and γ are the parameters of the underlying Weibull distribution, are a solution of the likelihood equations,

$$\frac{\partial \ell(\tilde{\theta})}{\partial \lambda} = \sum_{i=1}^{m}\frac{t_i^{\hat{\gamma}} e^{-\hat{\lambda} t_i^{\hat{\gamma}}} - t_{i-1}^{\hat{\gamma}} e^{-\hat{\lambda} t_{i-1}^{\hat{\gamma}}}}{e^{-\hat{\lambda} t_{i-1}^{\hat{\gamma}}} - e^{-\lambda t_i^{\gamma}}} X_i - t_m^{\hat{\gamma}} X_{m+1} = 0,$$

and

$$\frac{\partial \ell(\tilde{\theta})}{\partial \gamma} = -\hat{\lambda}\sum_{i=1}^{m} t_{i-1}^{\hat{\gamma}}(\ell n t_{i-1})X_i + \hat{\lambda}\sum_{i=1}^{m}\frac{t_i^{\hat{\gamma}}(\ell n t_i) - t_{i-1}^{\hat{\gamma}}(\ell n t_{i-1})}{e^{\hat{\lambda}(t_i^{\hat{\gamma}} - t_{i-1}^{\hat{\gamma}})} - 1} X_i - \hat{\lambda} t_m^{\hat{\gamma}}(\ell n t_m)X_{m+1} = 0.$$

These are two non-linear equations in $\hat{\lambda}$ and $\hat{\gamma}$ which can be solved by using an iterative procedure such as Newton-Raphson method. Even in the special case $t_i = it_0$, there is no explicit solution. The asymptotic variance-covariance matrix is given by the inverse of the matrix. using Theorem 3.1,

$$nA'A = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = n\sum_{i=1}^{m} \frac{\left[t_i^{\gamma} e^{-\lambda t_i^{\gamma}} - t_{i-1}^{\gamma} e^{-\lambda t_{i-1}^{\gamma}}\right]^2}{\left[e^{-\lambda t_{i-1}^{\gamma}} - e^{-\lambda t_i^{\gamma}}\right]} + nt_m^{2\gamma} e^{-\lambda t_m^{\gamma}},$$

$$S_{22} = n\lambda^2 \sum_{i=1}^{m} \frac{\left[t_i^{\gamma}(\ell n\, t_i) - t_{i-1}^{\gamma}(\ell n\, t_{i-1})\right]^2}{\left[e^{\lambda t_i^{\gamma}} - e^{\lambda t_{i-1}^{\gamma}}\right]},$$

and

$$S_{12} = S_{21} = n\lambda \sum_{i=1}^{m} \frac{(t_i^{\gamma} \ell n\, t_i - t_{i-1}^{\gamma} \ell n\, t_{i-1})(t_i^{\gamma} - t_{i-1}^{\gamma})}{\left[e^{\lambda t_i^{\gamma}} - e^{\lambda t_{i-1}^{\gamma}}\right]}.$$

6.3 Maximum Likelihood Estimation for the Lognormal Distribution

The probability density function of Lognormal distribution is given by

$$f_T(t) = \begin{cases} \dfrac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\ell nt - \mu)^2} & t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The parameter vector is $\tilde{\theta} = (\mu, \sigma^2)$ and the parameter space Θ is identified as $R \times R^+ = \left\{\tilde{\theta} = (\mu, \sigma^2);\ -\infty < \mu < \infty,\ \sigma > 0\right\}$. The random variables X_1, X_2, ..., X_m, X_{m+1} have a multinomial distribution with probabilities,

$$p_i(\tilde{\theta}) = \Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right),\ i = 1,\ 2,\ \ldots,\ m,$$

and

$$p_{m+1}(\tilde{\theta}) = 1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Likelihood of the data:

$$L(\tilde{\theta}) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right) \right]^{X_i} \left[1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right) \right]^{(n - \sum_{i=1}^{m} X_i)}.$$

Loglikelihood of the data:

$$\ell(\tilde{\theta}) = \text{const.} + \sum_{i=1}^{m} X_i \, \ell n \left[\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right) \right] + X_{m+1} \ell n \left(1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right) \right).$$

First-order derivatives of the loglikelihood:

$$\frac{\partial \ell(\tilde{\theta})}{\partial \mu} = \sum_{i=1}^{m} \frac{\phi(\ell nt_{i-1}) - \phi(\ell nt_i)}{\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right)} X_i + \frac{\phi(\ell nt_m)}{1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right)} X_{m+1},$$

and

$$\frac{\partial \ell(\tilde{\theta})}{\partial \sigma^2} = \sum_{i=1}^{m} \frac{\frac{(\ell nt_{i-1} - \mu)}{2\sigma^2} \phi(\ell nt_{i-1}) - \frac{(\ell nt_i - \mu)}{2\sigma^2} \phi(\ell nt_i)}{\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right)} X_i + \frac{\frac{(\ell nt_m - \mu)\phi(\ell nt_m)}{2\sigma^2}}{1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right)} X_{m+1},$$

where $\phi(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2}$.

The maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2, respectively, are found by equating the first-order derivatives to zero. The resulting equations are non-linear which can be solved by using an iterative procedure such as Newton-Raphson method. The asymptotic variance-covariance matrix is given by the inverse of the matrix,

$$nA'A = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})}\left[\frac{\partial p_i(\tilde{\theta})}{\partial \mu}\right]^2$$

$$= \sum_{i=1}^{m} \frac{\left[\phi(\ell nt_{i-1}) - \phi(\ell nt_i)\right]^2}{\left[\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right)\right]^2} + \frac{\phi^2(\ell nt_m)}{\left[1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right)\right]^2},$$

$$S_{22} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})}\left[\frac{\partial p_i(\tilde{\theta})}{\partial \sigma^2}\right]^2$$

$$= \sum_{i=1}^{m} \frac{\left[\frac{(\ell nt_{i-1} - \mu)}{2\sigma^2}\phi(\ell nt_{i-1}) - \frac{(\ell nt_i - \mu)}{2\sigma^2}\phi(\ell nt_i)\right]^2}{\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right)} + \frac{\left[\frac{(\ell nt_m - \mu)}{2\sigma^2}\phi(\ell nt_m)\right]^2}{1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right)},$$

and

$$S_{12} = S_{21} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})} \cdot \frac{\partial p_i(\tilde{\theta})}{\partial \mu} \cdot \frac{\partial p_i(\tilde{\theta})}{\partial \sigma^2}$$

$$= \sum_{i=1}^{m} \frac{\left[\phi(\ell nt_{i-1}) - \phi(\ell nt_i)\right]\left[\frac{(\ell nt_{i-1} - \mu)}{2\sigma^2}\phi(\ell nt_{i-1}) - \frac{(\ell nt_i - \mu)}{2\sigma^2}\phi(\ell nt_i)\right]}{\Phi\left(\frac{\ell nt_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell nt_{i-1} - \mu}{\sigma}\right)}$$

$$+ \frac{\frac{(\ell nt_m - \mu)}{2\sigma^2}\phi^2(\ell nt_m)}{1 - \Phi\left(\frac{\ell nt_m - \mu}{\sigma}\right)}.$$

6.4 Maximum Likelihood Estimation for the Logistic Distribution

The probability density function of Logistic distribution is given by

$$f_T(t) = \begin{cases} \dfrac{\lambda\gamma t^{\gamma-1}}{(1+\lambda t^{\gamma})^2} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The parameter vector is $\tilde{\theta} = (\lambda, \gamma)$, and the parameter space Θ is identified as $R^+ \times R^+ = \left\{\tilde{\theta} = (\lambda, \gamma);\ \lambda > 0,\ \sigma > 0\right\}$. The random variables X_1, X_2, ..., X_m, X_{m+1} have a multinomial distribution with probabilities,

$$p_i(\tilde{\theta}) = P(t_{i-1} < T < t_i),$$
$$= \frac{1}{1+\lambda t_i^{\gamma}} - \frac{1}{1+\lambda t_{i-1}^{\gamma}}, \quad i = 1,\ 2,\ \ldots,\ m,$$

and

$$p_{m+1}(\tilde{\theta}) = P(T > t_m)$$
$$= \frac{1}{1+\lambda t_m^{\gamma}}.$$

Likelihood of the data:

$$L(\tilde{\theta}) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[\frac{1}{1+\lambda t_{i-1}^{\gamma}} - \frac{1}{1+\lambda t_i^{\gamma}}\right]^{X_i} \left[\frac{1}{1+t_m^{\gamma}}\right]^{(n-\sum_{i=1}^{m} X_i)}$$

$$= \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[\frac{\lambda t_i^{\gamma} - \lambda t_{i-1}^{\gamma}}{(1+\lambda t_i^{\gamma})(1+\lambda t_{i-1}^{\gamma})}\right]^{X_i} \left[\frac{1}{1+t_m^{\gamma}}\right]^{X_{m+1}}.$$

Loglikelihood of the data:

$$\ell(\tilde{\theta}) = \text{const.} + \sum_{i=1}^{m} X_i\ \ell n\left[\lambda t_i^{\gamma} - \lambda t_{i-1}^{\gamma}\right] - \sum_{i=1}^{m} X_i\left[\ell n(1+\lambda t_i^{\gamma}) + \ell n(1+\lambda t_{i-1}^{\gamma})\right]$$
$$-X_{m+1}\ell n(1+\ell n(\lambda t_m^{\gamma})).$$

First-order derivatives of the loglikelihood:

$$\frac{\partial \ell(\tilde{\theta})}{\partial \lambda} = \sum_{i=1}^{m}\left[\frac{1}{\lambda} - \frac{t_i^{\gamma}}{1+\lambda t_i^{\gamma}} - \frac{t_{i-1}^{\gamma}}{1+\lambda t_{i-1}^{\gamma}}\right] X_i - \frac{t_m^{\gamma}}{1+\lambda t_m^{\gamma}} X_{m+1},$$

and

$$\frac{\partial\ell(\tilde{\theta})}{\partial\gamma} = \sum_{i=1}^{m} \frac{t_i^{\gamma}(\ell n t_i) - t_{i-1}^{\gamma}(\ell n t_{i-1})}{t_i^{\gamma} - t_{i-1}^{\gamma})} X_i - \sum_{i=1}^{m} \left[\frac{\lambda t_i^{\gamma}(\ell n t_i)}{1+\lambda t_i^{\gamma}} + \frac{\lambda t_{i-1}^{\gamma}(\ell n t_{i-1})}{1+\lambda t_{i-1}^{\gamma}} \right] X_i$$

$$- \frac{\lambda t_m^{\gamma}(\ell n t_m)}{1+\lambda t_m^{\gamma}} X_{m+1}.$$

The maximum likelihood estimators $\hat{\lambda}$ and $\hat{\gamma}$ of λ and γ, respectively, are found by equating the first-order derivatives to zero. The resulting equations are non-linear which can be solved by using an iterative procedure such as Newton-Raphson method. The asymptotic variance-covariance matrix is given by the inverse of the matrix,

$$nA'A = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$S_{11} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})} \left[\frac{\partial p_i(\tilde{\theta})}{\partial\lambda} \right]^2$$

$$= \sum_{i=1}^{m} \frac{\left[\dfrac{t_{i-1}^{\gamma}}{\left[1+\lambda t_{i-1}^{\gamma}\right]^2} - \dfrac{t_i^{\gamma}}{\left[1+\lambda t_i^{\gamma}\right]^2} \right]^2}{\left[\dfrac{1}{1+\lambda t_i^{\gamma}} - \dfrac{1}{1+\lambda t_{i-1}^{\gamma}} \right]} + \frac{t_m^{2\gamma}}{\left[1+\lambda t_m^{\gamma}\right]^3},$$

$$S_{22} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})} \left[\frac{\partial p_i(\tilde{\theta})}{\partial \gamma} \right]^2$$

$$= \sum_{i=1}^{m} \frac{\left[\dfrac{\lambda t_{i-1}^{\gamma} \ell n t_{i-1}}{\left[1+\lambda t_{i-1}^{\gamma}\right]^2} - \dfrac{\lambda t_i^{\gamma} \ell n t_i}{\left[1+\lambda t_i^{\gamma}\right]^2} \right]^2}{\left[\dfrac{1}{1+\lambda t_i^{\gamma}} - \dfrac{1}{1+\lambda t_{i-1}^{\gamma}} \right]} + \frac{\lambda^2 t_m^{2\gamma} (\ell n t_m)}{\left[1+\lambda t_m^{\gamma}\right]^3},$$

and

$$S_{12} = S_{21} = \sum_{i=1}^{m+1} \frac{1}{p_i(\tilde{\theta})} \cdot \frac{\partial p_i(\tilde{\theta})}{\partial \lambda} \cdot \frac{\partial p_i(\tilde{\theta})}{\partial \gamma}$$

$$= \sum_{i=1}^{m} \frac{\left[\dfrac{t_{i-1}^{\gamma}}{\left[1+\lambda t_{i-1}^{\gamma}\right]^2} - \dfrac{t_i^{\gamma}}{\left[1+\lambda t_i^{\gamma}\right]^2} \right] \left[\dfrac{\lambda t_{i-1}^{\gamma} \ell n t_{i-1}}{\left[1+\lambda t_{i-1}^{\gamma}\right]^2} - \dfrac{\lambda t_i^{\gamma} \ell n t_i}{\left[1+\lambda t_i^{\gamma}\right]^2} \right]}{\left[\dfrac{1}{1+\lambda t_i^{\gamma}} - \dfrac{1}{1+\lambda t_{i-1}^{\gamma}} \right]} + \frac{\lambda t_m^{2\gamma} (\ell n t_m)}{\left[1+\lambda t_m^{\gamma}\right]^3}.$$

6.5 Maximum Likelihood Estimation for the Pareto Distribution

The probability density function of Pareto distribution is given by

$$f_T(t) = \begin{cases} \dfrac{\gamma a^{\gamma}}{t^{\gamma+1}} & t \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma > 0$ is the unknown parameter of the model. Assume that $a > 0$ is known. In this case, the first inspection interval is $(a, t_1]$. The random variables $X_1, X_2, \ldots, X_m, X_{m+1}$ have a multinomial distribution with probabilities,

$$p_i = P(t_{i-1} < T \leq t_i),$$

$$= \left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}, \; i = 1, 2, \ldots, m,$$

$$p_{m+1} = P(T > t_m)$$
$$= \left(\frac{a}{t_m}\right)^{\gamma}.$$

Likelihood of the data:

$$L(\gamma) = \frac{n!}{\prod_{i=1}^{m+1} X_i!} \prod_{i=1}^{m} \left[\left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}\right]^{X_i} \left[\left(\frac{a}{t_m}\right)^{\gamma}\right]^{(n-\sum_{i=1}^{m} X_i)}$$

Loglikelihood of the data:

$$\ell(\gamma) = \text{const.} + \sum_{i=1}^{m} X_i \,\ell n\left[\left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}\right] + \gamma X_{m+1} \ell n\left[\frac{a}{t_m}\right]$$

First-order derivatives of the loglikelihood:

$$\frac{\partial \ell(\gamma)}{\partial \gamma} = \sum_{i=1}^{m} \frac{\left(\frac{a}{t_{i-1}}\right)^{\gamma} \ell n\left(\frac{a}{t_{i-1}}\right) - \left(\frac{a}{t_i}\right)^{\gamma} \ell n\left(\frac{a}{t_i}\right)}{\left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}} X_i + \left[\ell n\left(\frac{a}{t_m}\right)\right] X_{m+1}.$$

The maximum likelihood estimator $\hat{\gamma}$ of γ, is found by equating the first-order derivative to zero. The resulting equation is non-linear, which can be solved by using an iterative procedure such as Newton-Raphson method. The asymptotic variance of $\hat{\gamma}$ is given by the reciprocal of the information number $I(\gamma)$,

$$I(\gamma) = \sum_{i=1}^{m} \frac{\left[\left(\frac{a}{t_{i-1}}\right)^{\gamma} \ell n\left(\frac{a}{t_{i-1}}\right) - \left(\frac{a}{t_i}\right)^{\gamma} \ell n\left(\frac{a}{t_i}\right)\right]^2}{\left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}}.$$

6.6 Maximum Likelihood Estimation for the Extreme Value Distribution

The probability density function of Extreme value distribution is given by

$$f_X(x) = \gamma\, e^{\gamma(x-\theta)}\, e^{-e^{\gamma(x-\theta)}}, \qquad -\infty < x < \infty.$$

The parameter vector is $\tilde{\theta} = (\theta, \gamma)$, and the parameter space Θ is identified as $R \times R^+ = \left\{\tilde{\theta} = (\theta, \gamma);\ -\infty < \theta < \infty,\ \gamma > 0\right\}$. Let $T = e^X$. It can be shown that the probability density function of T is given by

$$f_T(t) = \lambda \gamma t^{\gamma-1} e^{-\lambda t^{\gamma-1}}$$

where $\lambda = e^{-\gamma\theta}$, which is the probability density function of the Weibull distribution with parameters γ and λ. Therefore, to find the maximum likelihood estimators for the parameters of the Extreme value distribution, we fit the Weibull distribution to the $T=e^X$ variable. The estimator $\hat{\gamma}$ of γ is equal to $\hat{\gamma}$, and the estimator $\hat{\theta}$ of θ is equal to $(-\ell n \hat{\lambda}) / \hat{\gamma}$. By using the Δ-method, the asymptotic variances of $\hat{\gamma}$ and $\hat{\theta}$ are given by

$$\text{asy}\,\text{var}(\hat{\gamma}) = S'_{22}$$

and

$$\text{as}\,\text{var}(\hat{\theta}) = \frac{1}{\lambda^2 \gamma^2} S'_{11} + \frac{(\ell n \lambda)^2}{\gamma^4} S'_{22} - 2 \frac{\ell n \lambda}{\lambda \gamma^3} S'_{12},$$

where

$$I(\tilde{\theta}) = \begin{bmatrix} S'_{11} & S'_{12} \\ & S'_{22} \end{bmatrix}$$

is the asymptotic variance-covariance of $\hat{\lambda}$ and $\hat{\gamma}$.

7. SOME RESULTS FROM MATRIX ALGEBRA

In this section, we establish some results in matrix algebra. These results will be used in the subsequent sections. Let the $m \times m$ symmetric matrix Σ have the following special structure:

$$\Sigma = \begin{pmatrix} \theta_1 & \theta_1 & \theta_1 & \theta_1 & \theta_1 & \cdots & \theta_1 & \theta_1 \\ & \theta_2 & \theta_2 & \theta_2 & \theta_2 & \cdots & \theta_2 & \theta_2 \\ & & \theta_3 & \theta_3 & \theta_3 & \cdots & \theta_3 & \theta_3 \\ & & & \theta_4 & \theta_4 & \cdots & \theta_4 & \theta_4 \\ & & & & \theta_5 & \cdots & \theta_5 & \theta_5 \\ & & & & & \cdots & \cdot & \cdot \\ & & & & & \cdots & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \theta_{(m-1)} & \theta_{(m-1)} \\ & & & & & & & \theta_m \end{pmatrix}.$$

(i) The determinant Δ of this matrix is given by

$$\prod_{i=1}^{m} (\theta_i - \theta_{i-1}),$$

with the convention that $\theta_0 = 0$.

(ii) The inverse matrix, $\Sigma^{-1} = (\sigma_{ij})$ of the matrix Σ is a tri-diagonal matrix and is given by

$$\sigma_{ii} = \frac{(\theta_{i+1} - \theta_{i-1})}{(\theta_i - \theta_{i-1})(\theta_{i+1} - \theta_i)}, \quad i = 1, 2, \ldots, m-1,$$

$$\sigma_{mm} = \frac{1}{(\theta_m - \theta_{m-1})},$$

and

$$\sigma_{i,i+1} = \sigma_{i+1,i} = -\frac{1}{(\theta_{i+1} - \theta_i)}.$$

(iii) Let

$$t_* = \begin{bmatrix} t_1 \\ t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_m \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

be $m \times 1$ vectors. Then

$$(t_*' \Sigma^{-1} t_*) = \sum_{i=1}^{m} t_i^2 \sigma_{ii} + 2 \sum_{i=2}^{m} t_{i-1} t_i \sigma_{i-1,i},$$

and

$$(t_*' \Sigma^{-1} Y) = \sum_{i=1}^{m} t_i y_i \sigma_{ii} + \sum_{i=2}^{m} (t_{i-1} y_i + t_i y_{i-1}) \sigma_{i-1,i}.$$

(iv) Let

$$t = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & t_m \end{bmatrix}$$

Then

$$(t' \Sigma^{-1} t) = \begin{bmatrix} \sum_{i=1}^{m} \sigma_{ii} + 2 \sum_{i=2}^{m} \sigma_{i-1,i} & \sum_{i=1}^{m} t_i \sigma_{ii} + \sum_{i=2}^{m} (t_{i-1} + t_i) \sigma_{i-1,i}) \\ & \sum_{i=1}^{m} t_i^2 \sigma_{ii} + 2 \sum_{i=2}^{m} t_{i-1} t_i \sigma_{i-1,i} \end{bmatrix},$$

and

$$(t' \Sigma^{-1} y) = \begin{bmatrix} \sum_{i=1}^{m} y_i \sigma_{ii} + \sum_{i=2}^{m} (y_{i-1} + y_i) \sigma_{i-1,i}) \\ \sum_{i=1}^{m} t_i y_i \sigma_{ii} + \sum_{i=2}^{m} (y_i t_{i-1} + t_i y_{i-1}) \sigma_{i-1,i}) \end{bmatrix}.$$

8. GRAPHICAL ANALYSIS OF INTERVAL-CENSORED DATA

Graphical methods are used to identify an appropriate distribution to model the distribution of a lifetime variable T. Suppose $t_1, t_2, \ldots, t_n$ are n independent realizations of T and no censoring occurred. One can plot the empirical distribution function of the data on probability papers such as Exponential, Gamma, Weibull, Lognormal, and Extreme value distribution probability paper. If the graph of the empirical distribution function of the data resembles a straight line on a particular probability paper, the corresponding probability model is used to model the distribution of T.

If the data are right-censored, the Kaplan-Meier estimate of the distribution function of T, which is an analogue of the empirical distribution, is used. The Kaplan-Meier estimate can be plotted on a variety of probability papers to identify the appropriate parametric distribution to model the distribution of T.

For interval-censored data, one can also employ probability papers to identify the underlying parametric model for the lifetime distribution. Nelson (1982) has demonstrated the use of the Weibull probability paper for plotting interval-censored data. He indicated that Exponential probability paper, Lognormal probability paper, and Extreme value probability paper can be used analogously to plot interval-censored data. To implement the procedure of Nelson, one needs to create all these probability papers before plotting the data.

In many statistical software packages such as STATPAC (Nelson *et al.*, 1978) and SAS (1994), creation of these special probability papers is inherent as a part of model building endeavor. In this section, we want to demonstrate how regular graph paper can be used to plot interval-censored data. We bring Logistic and Pareto distributions into our ambit in addition to the distributions considered by Nelson (1982).

We now spell out the problem more explicitly. Suppose we want to estimate the lifetime distribution T of a product. The sampling scheme that is in focus in this section runs as follows. Select a sample of n units of the product and set them to work. Inspect the units at times $t_1, t_2, \ldots, t_m$, where $t_1 < t_2 < \ldots < t_m$ are fixed. Let X_i = # units failed during the interval $(t_{i-1}, t_i]$, i=1, 2, ..., m and X_{m+1} = # units failed during (t_m, ∞), where $(t_{i-1}, t_i] = (0, t_1]$ for i = 1. Using the data X_1, X_2,

$\dots, X_m$, X_{m+1}, appropriate graphical methods will be developed to identify the underlying parametric lifetime distribution. The parametric distributions considered are Exponential, Weibull, Lognormal, Pareto, Logistic and Extreme value distribution.

8.1 Graphical Analysis for the Exponential Distribution

The probability density function of Exponential distribution is given by

$$f_T(t) = \begin{cases} \theta e^{-\theta t} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is the unknown parameter of the model. Let

$$\begin{aligned} p_i &= P(t_{i-1} < T \leq t_i), \\ &= e^{-\theta t_{i-1}} - e^{-\theta t_i}, \quad i = 1, 2, \dots, m, \end{aligned}$$

and

$$\begin{aligned} p_{m+1} &= P(T > t_m) \\ &= e^{-\theta t_m}. \end{aligned}$$

Note that for any j, $j = 1, 2, \dots, m$,

$$\sum_{i=1}^{j} p_i = 1 - e^{-\theta t_j}.$$

Let

$$q_j = \sum_{i=j+1}^{m+1} p_i,$$

and

$$\begin{aligned} r_j &= \ell n(q_j) \\ &= \ell n(\sum_{i=j+1}^{m+1} p_i) \\ &= -\theta t_j, \quad j = 1, 2, \dots, m. \end{aligned}$$

Define

$$Y_j = \ell n(\sum_{i=j+1}^{m+1} \frac{X_i}{n}).$$

The joint distribution of $X_1, X_2, ..., X_m, X_{m+1}$ is multinomial $(n, p_1, p_2, ..., p_m, p_{m+1})$.

Let $X_j^* = \sum_{i=j+1}^{m+1} X_j$. Note that $Y_j = \ell n(\frac{X_j^*}{n})$. Observe that X_j^* is distributed as Binomial (n, q_j). Using the Δ-method (Rao, 1973, p. 426), we get

$$\begin{aligned} E(Y_j) &= E(\ell n(\frac{X_j^*}{n})) \\ &\cong \ell n(\frac{1}{n} n \sum_{i=j+1}^{m+1} p_i) \\ &= -\theta t_j, \end{aligned}$$

which can be written as

$$E(Y_j) \cong \beta t_j,$$

where $\beta = -\theta$, $j = 1, 2, ..., m$. It is now clear that the random variables $Y_1, Y_2, ..., Y_m$ conform to a linear model through the origin. This observation can be exploited as follows. Let $x_1, x_2, ..., x_m, x_{m+1}$ be the realizations of $X_1, X_2, ..., X_m, X_{m+1}$, respectively. Let $y_j = \ell n(\sum_{i=j+1}^{m+1} \frac{x_i}{n})$, $j = 1, 2, ..., m$. Plot (t_j, y_j), $j = 1, 2, ..., m$ on a regular graph paper. If the points lie more or less on a straight line, Exponential distribution for T is worth a try. If this is the case, the preliminary estimate of θ can be obtained from the graph. This estimate is given by the negative slope of the straight line.

8.2 Graphical Analysis for the Weibull Distribution

The probability density function of Weibull distribution is given by

$$f_T(t) = \begin{cases} \lambda \gamma t^{\gamma-1} e^{-\lambda t^\gamma} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ and $\gamma > 0$ are the unknown parameters of the model. Let

$$\begin{aligned} p_i &= P(t_{i-1} < T \leq t_i), \\ &= e^{-\lambda t_{i-1}^\gamma} - e^{-\lambda t_i^\gamma}, \quad i = 1, 2, ..., m, \end{aligned}$$

and

$$p_{m+1} = P(T > t_m)$$
$$= e^{-\lambda t_m^\gamma}.$$

Note that for any j, $j = 1, 2, ..., m$,

$$\sum_{i=1}^{j} p_i = 1 - e^{-\lambda t_j^\gamma}.$$

Let

$$q_j = \sum_{i=j+1}^{m+1} p_i,$$

and

$$r_j = \ell n(-\ell n(q_j)$$
$$= \ell n(-\ell n \sum_{i=j+1}^{m+1} p_i)$$
$$= \ell n\lambda + \gamma \ell n t_j, \quad j = 1, 2, ..., m.$$

Define

$$Y_j = \ell n(-\ell n \sum_{i=j+1}^{m+1} \frac{X_i}{n}).$$

The joint distribution of $X_1, X_2, ..., X_{m,}, X_{m+1}$ is multinomial $(n, p_1, p_2, ..., p_m, p_{m+1})$.

Let $X_j^* = \sum_{i=j+1}^{m+1} X_j$. Note that $Y_j = \ell n(-\ell n \frac{X_j^*}{n})$. Observe that X_j^* is distributed as Binomial (n, q_j). Using the Δ-method, we get

$$E(Y_j) = E(\ell n(-\ell n \frac{X_j^*}{n})$$
$$\cong \ell n(-\ell n \frac{1}{n} n \sum_{i=j+1}^{m+1} p_i)$$
$$= \ell n \lambda + \gamma \ell n t_j,$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = \ell n \lambda$, $\beta = \gamma$, and $s_j = \ell n\, t_j$, j = 1, 2, ...,m. It is now clear that the random variables Y_1, Y_2, .., Y_m conform to a linear model. This observation can be exploited as follows. Let x_1, x_2, ..., x_m, x_{m+1} be the realizations of X_1, X_2, ..., X_m, X_{m+1}, respectively. Let $y_j = \ell n(-\ell n \sum_{i=j+1}^{m+1} \frac{x_i}{n})$, j = 1, 2, ..., m. Plot $(\ell n t_j, y_j)$, j = 1, 2, ..., m on a regular graph paper. If the points lie more or less on a straight line, Weibull distribution for T is worth a try. If this is the case, preliminary estimates of λ and γ can be obtained from the graph. The shape parameter γ is estimated by the slope of the straight line, and the scale parameter λ is estimated by the exponent of the intercept of the straight line.

At this juncture, it is appropriate to spell out the essential difference between our approach and Nelson's approach in plotting interval-censored data. We and Nelson (1982) use the same relation

$$\ell n(-\ell n\, P(T > t_j)) = \ell n\lambda + \gamma \ell n\, t_j, \; j = 1, 2, ..., m.$$

Nelson (1982) takes the x-axis on the logarithmic scale, y-axis on the $\ell n(-\ell n)$ scale, and the interval-censored data in the form $(t_j, \hat{q}_j)$, j = 1, 2, ..., m is plotted on this specially created paper, where $\hat{q}_j = \sum_{i=j+1}^{m+1} \frac{x_i}{n}$. In our approach, we advocate plotting $(\ell n t_j, \ell n(-\ell n\, \hat{q}_j))$ directly on a standard graph paper. There are two distinct advantages our method of plotting has over the method of Nelson.

1. The data points $(\ell n t_j, \ell n(-\ell n\, \hat{q}_j))$, j = 1, 2, ..., m, can be plotted precisely on the standard graph paper. On the Weibull probability paper, the point $(t_j, \hat{q}_j)$ can only be identified approximately.
2. Preliminary estimates of the scale and shape parameters of the Weibull distribution are easier to obtain from the standard graph paper.

As an illustration, for the following data (Table 8.1) on 167 identical parts in a machine (Nelson, 1982, p. 415), the two plots are presented (Figures 8.1 and 8.2).

At certain ages, the parts were inspected to determine which had cracked since the last previous inspection. The inspection times are 6.12, 19.92, 29.64, 35.40, 39.72, 45.24, 52.32, and 63.48 months respectively. The data appear in Table 8.1; it shows the months, in service at the start and end of each inspection period and the number of cracked parts found in each period. For example, between 6.12 and 19.92 months, 16 parts cracked. Seventy three parts survived the latest inspection at 63.48 months.

Table 8.1
Part Cracking Data

Inspection Intervals	Number Cracked	Cumulative	$\hat{q}_j$	$1-\hat{q}_j$	$\ell n - \ell n \hat{q}_j$
(0.00, 6.12]	5	5	0.970	0.023	-3.493
(6.12, 19.92]	16	21	0.874	0.126	-2.007
(19.92, 29.64]	12	33	0.802	0.198	-1.513
(29.64, 35.40]	18	51	0.695	0.305	-1.009
(35.40, 39.72]	18	69	0.587	0.413	-0.629
(39.72, 45.24]	2	71	0.575	0.425	-0.591
(45.24, 52.32]	6	77	0.539	0.461	-0.481
(52.32, 63.48]	17	94	0.437	0.563	-0.189
(63.48, ∞)	73	167			
Total	167				

Figure 8.1

Estimation of the Weibull Distribution Parameters for the Part Cracking Data Using Standard Graph Paper

The preliminary estimates of the Weibull parameters from the standard graph paper are $\hat{\lambda} = 0.00452$ and $\hat{\gamma} = 1.636$.

Figure 8.2

Estimation of the Weibull Distribution Parameters for the Part Cracking Data Using Weibull Probability paper

The preliminary estimates of the Weibull parameters from the Weibull Probability paper are $\hat{\lambda} = 0.0017$ and $\hat{\gamma} = 1.49$.

8.3 Graphical Analysis for the Lognormal Distribution

The probability density function of the Lognormal distribution is given by

$$f_T(t) = \begin{cases} \dfrac{1}{t\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\ell n t-\mu)^2} & t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma > 0$ and $-\infty < \mu < \infty$ are the unknown parameters of the model. Let

$$\begin{aligned} p_i &= P(t_{i-1} < T \le t_i), \\ &= \Phi\left(\frac{\ell n t_i - \mu}{\sigma}\right) - \Phi\left(\frac{\ell n t_{i-1} - \mu}{\sigma}\right), \quad i = 1,\ 2,\ \ldots,\ m, \end{aligned}$$

and

$$\begin{aligned} p_{m+1} &= P(T > t_m) \\ &= 1 - \Phi\left(\frac{\ell n t_m - \mu}{\sigma}\right), \end{aligned}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Note that for any j, $j = 1, 2, \ldots, m$,

$$\sum_{i=1}^{j} p_i = \Phi\left(\frac{\ell n t_j - \mu}{\sigma}\right).$$

Let

$$q_j = \sum_{i=j+1}^{m+1} p_i = 1 - \Phi\left(\frac{\ell n t_j - \mu}{\sigma}\right),$$

and

$$\begin{aligned} r_j &= \Phi^{-1}(1 - q_j), \\ &= -\frac{\mu}{\sigma} + \frac{1}{\sigma} \ell n\, t_j, \quad i = 1,\ 2,\ \ldots,\ m. \end{aligned}$$

Define

$$Y_j = \Phi^{-1}(1 - \sum_{i=j+1}^{m+1} \frac{X_i}{n}).$$

The joint distribution of $X_1, X_2, \ldots, X_m, X_{m+1}$ is multinomial $(n, p_1, p_2, \ldots, p_m, p_{m+1})$.

Let $X_j^* = \sum_{i=j+1}^{m+1} X_j$. Note that $Y_j = \Phi^{-1}(1 - \frac{X_j^*}{n})$. Observe that X_j^* is distributed as Binomial (n, q_j). Using the Δ-method, we get

$$\begin{aligned} E(Y_j) &= E\Phi^{-1}(1 - \frac{X_j^*}{n}) \\ &\cong \Phi^{-1}(1 - \frac{1}{n} n \sum_{i=j+1}^{m+1} p_i) \\ &= -\frac{\mu}{\sigma} + \frac{1}{\sigma} \ell n t_j, \end{aligned}$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = -\mu/\sigma$, $\beta = 1/\sigma$, and $s_j = \ell n\, t_j$, $j = 1, 2, \ldots, m$. It is now clear that the random variables $Y_1, Y_2, \ldots, Y_m$ conform to a linear model. Let $x_1, x_2, \ldots, x_m, x_{m+1}$ be the realizations of $X_1, X_2, \ldots, X_m, X_{m+1}$, respectively. Let $y_j = \Phi^{-1}(1 - \sum_{i=j+1}^{m+1} \frac{x_i}{n})$, $j = 1, 2, \ldots, m$. Plot $(\ell n t_j, y_j)$, $j = 1, 2, \ldots, m$ on a regular graph paper. If the points lie more or less on a straight line, Lognormal distribution for T is a possible model. If this is the case, preliminary estimates of μ and σ can be obtained from the graph. The estimate $\hat{\sigma}$ of σ is given by $1/\beta$, and the estimate $\hat{\mu}$ of μ is given by $-\alpha/\beta$, where α and β are the intercept and slope, respectively, of the free hand straight line passing close to the points plotted.

8.4 Graphical Analysis for the Pareto Distribution

The probability density function of Pareto distribution is given by

$$f_T(t) = \begin{cases} \dfrac{\gamma a^{\gamma}}{t^{\gamma+1}} & t \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma > 0$ is the unknown parameter of the model. Assume that a is known In this case, the first inspection interval is $(a, t_1]$. Let

$$p_i = P(t_{i-1} < T \leq t_i),$$
$$= \left(\frac{a}{t_{i-1}}\right)^{\gamma} - \left(\frac{a}{t_i}\right)^{\gamma}, \ i = 1, 2, ..., m,$$

and

$$p_{m+1} = P(T > t_m)$$
$$= \left(\frac{a}{t_m}\right)^{\gamma}.$$

Note that for any j, $j = 1, 2, ..., m$,

$$\sum_{i=1}^{j} p_i = 1 - \left(\frac{a}{t_j}\right)^{\gamma}.$$

Let

$$q_j = \sum_{i=j+1}^{m+1} p_i = \left(\frac{a}{t_j}\right)^{\gamma},$$

and

$$r_j = \ell n\, q_j = \ell n\left(\sum_{i=j+1}^{m+1} p_i\right),$$
$$= \gamma \ell n\left(\frac{a}{t_j}\right), \ j = 1, 2, ..., m.$$

Define

$$Y_j = \ell n\left(\sum_{i=j+1}^{m+1} \frac{X_i}{n}\right).$$

The joint distribution of $X_1, X_2, \ldots, X_m, X_{m+1}$ is multinomial $(n, p_1, p_2, \ldots, p_m, p_{m+1})$.

Let $X_j^* = \sum_{i=j+1}^{m+1} X_j$. Note that $Y_j = \ell n \frac{X_j^*}{n}$. Observe that X_j^* is distributed as Binomial (n, q_j). Using the Δ-method, we get

$$
\begin{aligned}
E(Y_j) &= E(\ell n \frac{X_j^*}{n}) \\
&\cong \ell n(\frac{1}{n} n \sum_{i=j+1}^{m+1} p_i) \\
&= \gamma \ell n(\frac{a}{t_j}), \; j = 1, 2, \ldots, m.
\end{aligned}
$$

It is now clear that the random variables $Y_1, Y_2, \ldots, Y_m$ conform to a linear model through the origin.. Let $x_1, x_2, \ldots, x_m, x_{m+1}$ be the realizations of $X_1, X_2, \ldots, X_m, X_{m+1}$, respectively. Let $y_j = \ell n(\sum_{i=j+1}^{m+1} \frac{x_i}{n})$, $j = 1, 2, \ldots, m$. Plot $(\ell n(\frac{a}{t_j}), y_j)$, $j = 1, 2, \ldots, m$ on a regular graph paper. If the points lie more or less on a straight line, Pareto distribution is a possible model for T. If this is the case, preliminary estimate $\hat{\gamma}$ of γ can be obtained from the graph. This estimate is given by the slope of the straight line.

8.5 Graphical Analysis for the Logistic Distribution

The probability density function of Logistic distribution is given by

$$
f_T(t) = \begin{cases} \dfrac{\lambda \gamma t^{\gamma-1}}{(1+\lambda t^{\gamma})^2} & t \geq 0, \\ 0 & \text{otherwise,} \end{cases}
$$

where $\lambda > 0$ and $\gamma > 0$ are the unknown parameters of the model. Let

$$
\begin{aligned}
p_i &= P(t_{i-1} < T \leq t_i), \\
&= \frac{1}{1+\lambda t_{i-1}^{\gamma}} - \frac{1}{1+\lambda t_i^{\gamma}}, \quad i = 1, 2, \ldots, m,
\end{aligned}
$$

and

$$p_{m+1} = P(T > t_m)$$
$$= \frac{1}{1+\lambda t_m^{\gamma}}.$$

Note that for any j, $j = 1, 2, ..., m$,

$$\sum_{i=1}^{j} p_i = 1 - \frac{1}{1+\lambda t_j^{\gamma}}.$$

Let

$$q_j = \sum_{i=j+1}^{m+1} p_i = \frac{1}{1+\lambda t_j^{\gamma}},$$

and

$$r_j = \ell n \left(\frac{1-q_j}{q_j}\right) = \ell n \left(\frac{\sum_{i=1}^{j} p_i}{\sum_{i=j+1}^{m+1} p_i} \right) = \ell n \left(\frac{1}{\sum_{i=j+1}^{m+1} p_i} - 1 \right),$$
$$= \ell n\, \lambda + \gamma \ell n\, t_j, \quad j = 1, 2, ..., m.$$

Define

$$Y_j = \ell n \left(\frac{1}{\sum_{i=j+1}^{m+1} \frac{X_i}{n}} - 1 \right).$$

The joint distribution of $X_1, X_2, ..., X_m, X_{m+1}$ is multinomial $(n, p_1, p_2, ..., p_m, p_{m+1})$.

Let $X_j^* = \sum_{i=j+1}^{m+1} X_i$. Note that $Y_j = \ell n(\frac{n}{X_j^*} - 1)$. Observe that X_j^* is distributed as Binomial (n, q_j). Using the Δ-method, we get

$$E(Y_j) = E\ell n(\frac{n}{X_j^*} - 1)$$

$$\cong \ell n\left(\frac{n}{n\sum_{i=j+1}^{m+1} p_i} - 1\right)$$

$$= \ell n\lambda + \gamma \ell n t_j,$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = \ell n\, \lambda$, $\beta = \gamma$ and $s_j = \ell n\, t_j$, j = 1, 2, ...,m. It is now clear that the random variables $Y_1, Y_2, \ldots, Y_m$ conform to a linear model. We use this as follows. Let $x_1, x_2, \ldots, x_{m+1}$ be the realizations of $X_1, X_2, \ldots, X_m, X_{m+1}$, respectively. Let $y_j = \ell n\left(\frac{1}{\sum_{i=j+1}^{m+1} \frac{x_i}{n}} - 1\right)$, j =1, 2, ..., m. Plot $(\ell n t_j, y_j)$, j = 1, 2, ..., m on a regular graph paper. If the points lie more or less on a straight line, Logistic distribution for T is worth a try. If this is the case, preliminary estimates of γ and a can be obtained from the graph. The estimate $\hat{\gamma}$ of γ is given by β, and the estimate $\hat{\lambda}$ of λ is given by e^{α}, where α and β are the intercept and the slope, respectively, of the free hand straight line passing close to the points plotted.

8.6 Graphical Analysis for the Extreme Value Distribution

The probability density function of Extreme value distribution is given by

$$f_X(x) = \gamma e^{\gamma(x-\theta)} e^{-e^{\gamma(x-\theta)}}, \quad -\infty < x < \infty$$

where $-\infty < \theta < \infty$ and $\gamma > 0$ are the unknown parameters of the model. Let

$$T = e^X.$$

It can be shown that the probability density function of T is given by

$$f_T(t) = \lambda\gamma t^{\gamma-1} e^{-\lambda t^{\gamma-1}}$$

where $\lambda = e^{-\gamma\theta}$, which is the probability density function of the Weibull distribution with parameters γ and λ. Therefore, to estimate the parameters of the Extreme value distribution, we fit the Weibull distribution to the $T=e^X$ variable. The estimate $\hat{\gamma}$ of γ is equal to $\hat{\gamma}$, and the estimate $\hat{\theta}$ of θ is equal to $-\ell n\hat{\lambda} / \hat{\gamma}$.

9. LINEAR MODEL APPROACH TO INTERVAL-CENSORED DATA

Suppose we want to estimate the lifetime distribution T of a product. The sampling scheme that is the general periodic inspection plan outlined in Section 8. Using the data X_1, X_2, ..., X_m, X_{m+1}, the main focus in this section is on the estimation of the parameters of the underlying distribution of T following the Linear Model Approach. This method of estimation is simpler than the method of maximum likelihood. These two methods give asymptotically consistent estimators. We examine how they perform in small samples. The parametric distributions considered are Exponential, Weibull, Lognormal, Pareto, Logistic, and Extreme value distributions.

9.1 Linear Model Approach for the Exponential Distribution

The starting point is the collection of random variables,

$$Y_j = \ell n(\sum_{i=j+1}^{m+1} \frac{X_i}{n}), \quad i = 1,\ 2,\ \ldots,\ m.$$

Using the Δ-method, we have noted that

$$\begin{aligned} E(Y_j) &= E(\ell n(\frac{X_j^*}{n})) \\ &\cong \ell n(\frac{1}{n} nq_j) \\ &= -\theta\, t_j, \end{aligned}$$

which can be written as

$$E(Y_j) \cong \beta t_j,$$

where $\beta = -\theta$, $j = 1, 2, ..., m$. Using the Δ-method again, it can be shown that

$$\begin{aligned}
\mathrm{Var}(Y_j) &= \mathrm{Var}(\ell n(\frac{X_j^*}{n})) \\
&\cong \mathrm{Var}(X_j^*)\left[\frac{d}{dX_j^*}\ell n(\frac{X_j^*}{n})\right]^2_{X_j^*=EX_j^*} \\
&= \frac{1-q_j}{nq_j},
\end{aligned}$$

and for $j < k$,

$$\begin{aligned}
\mathrm{Cov}(Y_j, Y_k) &= \mathrm{Cov}(\ell n(\frac{X_j^*}{n}), \ell n(\frac{X_k^*}{n})) \\
&\cong \mathrm{Cov}(X_j^*, X_k^*)\left[\frac{d}{dX_j^*}\ell n(\frac{X_j^*}{n})\right]\left[\frac{d}{dX_k^*}\ell n(\frac{X_k^*}{n})\right]_{\substack{X_k^*=EX_k^* \\ X_j^*=EX_j^*}} \\
&= \mathrm{Cov}(\sum_{i=j+1}^{m+1} X_j, \sum_{i=k+1}^{m+1} X_k)\left[\frac{d}{dX_j^*}\ell n(\frac{X_j^*}{n})\right]\left[\frac{d}{dX_k^*}\ell n(\frac{X_k^*}{n})\right]_{\substack{X_k^*=EX_k^* \\ X_j^*=EX_j^*}} \\
&= \left[\mathrm{Var}(\sum_{i=k+1}^{m+1} X_i) + \sum_{r=j+1}^{k}\sum_{s=k+1}^{m+1}\mathrm{Cov}(X_r, X_s)\right]\left[\frac{d}{dX_j^*}\ell n(\frac{X_j^*}{n})\right]\times \\
&\qquad \left[\frac{d}{dX_k^*}\ell n(\frac{X_k^*}{n})\right]_{\substack{X_k^*=EX_k^* \\ X_j^*=EX_j^*}} \\
&= \left[nq_k(1-q_k) + \sum_{r=j+1}^{k}\sum_{s=k+1}^{m+1} -np_r p_s\right]\frac{1}{nq_j}\frac{1}{nq_k} \\
&= \frac{1-q_j}{nq_j}.
\end{aligned}$$

In general, for $j \neq k$,

$$\mathrm{Cov}(Y_j,\ Y_k) = \frac{1-q_s}{nq_s},$$

where $s = \mathrm{Min}\ (j, k)$.

The variances and covariances of Y_i's have the following properties:

$$\mathrm{Var}(Y_1) = \mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(Y_1, Y_3) = ... = \mathrm{Cov}(Y_1, Y_m)\ ;$$

$$\mathrm{Var}(Y_2) = \mathrm{Cov}(Y_2, Y_3) = \mathrm{Cov}(Y_2, Y_4) = \ldots = \mathrm{Cov}(Y_2, Y_m)\ ;$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\mathrm{Var}(Y_{m-1}) = \mathrm{Cov}(Y_{m-1}, Y_m).$$

Let us introduce a special notation exploiting the special structure of these variances and covariances. Let

$$\theta_k = \mathrm{Var}(Y_k) = \frac{p_k}{nq_k},\ k = 1,\ 2,\ \ldots,\ m.$$

These computations can be summarized as follows:

$$E(Y) = E\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \cdot \\ \cdot \\ Y_m \end{pmatrix} = \begin{pmatrix} -t_1 \\ -t_2 \\ -t_3 \\ \cdot \\ \cdot \\ -t_m \end{pmatrix}\theta = t\beta,$$

the dispersion matrix Σ of Y is given by

$$\Sigma = \begin{pmatrix}
\theta_1 & \theta_1 & \theta_1 & \theta_1 & \theta_1 & \cdot & \cdot & \cdot & \theta_1 & \theta_1 \\
 & \theta_2 & \theta_2 & \theta_2 & \theta_2 & \cdot & \cdot & \cdot & \theta_2 & \theta_2 \\
 & & \theta_3 & \theta_3 & \theta_3 & \cdot & \cdot & \cdot & \theta_3 & \theta_3 \\
 & & & \theta_4 & \theta_4 & \cdot & \cdot & \cdot & \theta_4 & \theta_4 \\
 & & & & \theta_5 & \cdot & \cdot & \cdot & \theta_5 & \theta_5 \\
 & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & \cdot & \cdot & \cdot \\
 & & & & & & & & \theta_{(m-1)} & \theta_{(m-1)} \\
 & & & & & & & & & \theta_m
\end{pmatrix}.$$

Let $\Sigma^{-1} = (\sigma_{ij})$. In Section 6, we have identify the inverse Σ^{-1} to be a tri-diagonal matrix. If Σ is known, the best linear unbiased estimator $\hat{\theta}_n$ of θ is given by

$$
\hat{\theta}_n = (t'\Sigma^{-1}t)^{-1}t'\Sigma^{-1}Y
$$

$$
= \frac{\sum_{i=1}^{m} t_i y_i \sigma_{ii} + \sum_{i=2}^{m} (t_{i-1}y_i + t_i y_{i-1})\sigma_{i-1,i}}{\sum_{i=1}^{m} t_i^2 \sigma_{ii} + 2\sum_{i=2}^{m} (t_{i-1}t_i)\sigma_{i-1,i}}.
$$

Since Σ is unknown, one can estimate Σ consistently by $\hat{\Sigma}$, where

$$
\hat{\Sigma} = \begin{pmatrix}
\hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \cdots & \hat{\theta}_1 & \hat{\theta}_1 \\
 & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & \cdots & \hat{\theta}_2 & \hat{\theta}_2 \\
 & & \hat{\theta}_3 & \hat{\theta}_3 & \hat{\theta}_3 & \cdots & \hat{\theta}_3 & \hat{\theta}_3 \\
 & & & \hat{\theta}_4 & \hat{\theta}_4 & \cdots & \hat{\theta}_4 & \hat{\theta}_4 \\
 & & & & \hat{\theta}_5 & \cdots & \hat{\theta}_5 & \hat{\theta}_5 \\
 & & & & & \ddots & \vdots & \vdots \\
 & & & & & & \hat{\theta}_{(m-1)} & \hat{\theta}_{(m-1)} \\
 & & & & & & & \hat{\theta}_m
\end{pmatrix},
$$

and

$$
\hat{\theta}_j = \frac{n - x_j^*}{n x_j^*}.
$$

The modified estimator $\hat{\theta}_n$ of θ is given by

$$
\hat{\theta}_n = (t'\hat{\Sigma}^{-1}t)^{-1}t'\hat{\Sigma}^{-1}Y
$$

$$
= \frac{\sum_{i=1}^{m} t_i y_i \hat{\sigma}_{ii} + \sum_{i=2}^{m} (t_{i-1}y_i + t_i y_{i-1})\hat{\sigma}_{i-1,i}}{\sum_{i=1}^{m} t_i^2 \hat{\sigma}_{ii} + 2\sum_{i=2}^{m} (t_{i-1}t_i)\hat{\sigma}_{i-1,i}}.
$$

where

$$
\hat{\Sigma}^{-1} = (\hat{\sigma}_{ij}).
$$

Some useful facts emerge from these deliberations.

1. We have an explicit formula for the estimator of θ. The maximum likelihood method does not provide an explicit formula for the maximum likelihood estimator of θ.
2. The new estimator is also consistent. For

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} \hat{\theta}_n &= \operatorname*{plim}_{n\to\infty}(t'\hat{\Sigma}^{-1}t)^{-1}t'\hat{\Sigma}^{-1}Y \\
&= (t' \operatorname*{plim}_{n\to\infty} \hat{\Sigma}^{-1}t)^{-1}t'(\operatorname*{plim}_{n\to\infty} \hat{\Sigma}^{-1})(\operatorname*{plim}_{n\to\infty} Y), \text{ (Slutsky's theorem)} \\
&= (t'\Sigma^{-1}t)^{-1}t'\Sigma^{-1}t\beta \\
&= \beta = \theta.
\end{aligned}$$

In one special case, the maximum likelihood method provides an explicit formula for its estimator. This estimator differs from the linear model estimator. Let $t_i = it_0$, $i = 1, 2, \ldots, m$ for some $t_0 > 0$.

Linear model estimator:

$$\hat{\theta}_n = \frac{\sum_{i=1}^{m} iy_i\hat{\sigma}_{ii} + \sum_{i=2}^{m} ((i-1)y_i + iy_{i-1})\hat{\sigma}_{i-1,i}}{t_0\sum_{i=1}^{m} i^2\hat{\sigma}_{ii} + 2\sum_{i=2}^{m} (i(i-1))\hat{\sigma}_{i-1,i}}.$$

Maximum likelihood estimator:

$$\hat{\theta}_n = \frac{1}{t_0}\ell n\left(\frac{mn + \sum_{i=1}^{m}(i-m)X_i}{mn + \sum_{i=1}^{m}(i-1-m)X_i}\right).$$

In the particular case $m = 2$, one can write down the estimators in terms of the data X_1, X_2, and X_3.

Linear model estimator:

$$\hat{\theta}_n = \frac{1}{t_0}\left(\frac{X_1X_3}{(X_1+X_2)(X_2+X_3)}\ell n\left(\frac{X_2+X_3}{X_3}\right) - \frac{nX_2}{(X_1+X_2)(X_2+X_3)}\ell n\left(\frac{X_2+X_3}{n}\right)\right).$$

Maximum likelihood estimator:

$$\hat{\theta}_n = \frac{1}{t_0}\ell n\frac{2n - X_1}{2n - 2X_1 - X_2}.$$

A warning should be sounded at this juncture. In practice, especially if n is small, the observed values of some X_i's could be zero. In this case, the estimated dispersion matrix $\hat{\Sigma}$ is singular. The linear model estimator does not exist. The maximum likelihood is also not free of problems. The iterative procedure used to estimate the parameters fails to converge. One can note that $P(X_i = 0)$ converges to 0 exponentially fast as $n \to \infty$. Consequently, the occurrence of the event $\{ X_i = 0\}$ is unlikely.

9.2 Linear Model Approach for the Weibull Distribution

We begin with the random variables,

$$Y_j = \ell n(-\ell n \sum_{i=j+1}^{m+1} \frac{X_i}{n}).$$

Using the Δ-method, we have noted that

$$E(Y_j) \cong \ell n\, \lambda + \gamma \ell n t_j,$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = \ell n\, \lambda$, $\beta = \gamma$, and $s_j = \ell n\, t_j$, $j = 1, 2, ..., m$. It is now clear that the random variables $Y_1, Y_2, ..., Y_m$ conform to a linear model. Using the Δ-method again, it can be shown that

$$\begin{aligned} Var(Y_j) &= Var(\ell n(-\ell n(\frac{X_j^*}{n})) \\ &\cong Var(X_j^*)\left[\frac{d}{dX_j^*}\ell n(-\ell n(\frac{X_j^*}{n})\right]^2_{X_j^* = EX_j^*} \\ &= \frac{1-q_j}{nq_j(\ell n\, q_j)^2}, \end{aligned}$$

and for $j < k$,

$$
\begin{aligned}
\mathrm{Cov}(Y_j, Y_k) &= \mathrm{Cov}(\ell n(-\ell n(\frac{X_j^*}{n}), \ell n(-\ell n(\frac{X_k^*}{n})) \\
&\cong \mathrm{Cov}(X_j^*, X_k^*)\left[\frac{d}{dX_j^*}\ell n(-\ell n(\frac{X_j^*}{n})\right]\left[\frac{d}{dX_k^*}\ell n(-\ell n(\frac{X_k^*}{n})\right]_{\substack{X_k^*=EX_k^* \\ X_j^*=EX_j^*}} \\
&= \frac{1-q_j}{nq_j(\ell nq_j)(\ell nq_k)}.
\end{aligned}
$$

In general, for $j \neq k$,

$$\mathrm{Cov}(Y_j,\ Y_k) = \frac{1-q_s}{nq_s(\ell nq_j)(\ell nq_k)},$$

where $s = \mathrm{Min}\ (j, k)$.

Let us introduce a special notation exploiting the special structure of these variances and covariances. Let

$$\theta_j = \frac{1-q_j}{nq_j},$$

and

$$\pi_j = \frac{1}{\ell nq_j}.$$

Then,

$$\mathrm{Var}(Y_j) = \theta_j \pi_j,\ j = 1,\ 2,\ \ldots,\ m,$$

and for $j \neq k$,

$$\mathrm{Cov}(Y_j, Y_k) = \theta_s \pi_j \pi_k,$$

where $s = \mathrm{Min}\ (j, k)$.

These computations can be summarized as follows:

$$E(Y) = E(\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ Y_m \end{pmatrix}) = \begin{pmatrix} 1 & \ell nt_1 \\ 1 & \ell nt_2 \\ 1 & \ell nt_3 \\ . & . \\ . & . \\ 1 & \ell nt_m \end{pmatrix} . \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S\beta,$$

the dispersion matrix Σ of Y is given by

$$\Sigma = \begin{bmatrix} \pi_1^2\theta_1 & \pi_1\pi_2\theta_1 & \pi_1\pi_3\theta_1 & \pi_1\pi_4\theta_1 & \cdot\ \cdot\ \cdot & \pi_1\pi_{(m-1)}\theta_1 & \pi_1\pi_m\theta_1 \\ & \pi_2^2\theta_2 & \pi_2\pi_3\theta_2 & \pi_2\pi_4\theta_2 & \cdot\ \cdot\ \cdot & \pi_2\pi_{(m-1)}\theta_2 & \pi_2\pi_m\theta_2 \\ & & \pi_3^2\theta_3 & \pi_3\pi_4\theta_3 & \cdot\ \cdot\ \cdot & \pi_3\pi_{(m-1)}\theta_3 & \pi_3\pi_m\theta_3 \\ & & & \pi_4^2\theta_4 & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \pi_{(m-1)}^2\theta_{(m-1)} & \pi_{(m-1)}\pi_m\theta_{(m-1)} \\ & & & & & & \pi_m^2\theta_m \end{bmatrix}.$$

Let $\Sigma^{-1} = (\sigma_{ij})$. The inverse Σ^{-1} is a tri-diagonal matrix. If Σ is known, the best linear unbiased estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively, are given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (S'\Sigma^{-1}S)^{-1}S'\Sigma^{-1}Y.$$

Since Σ is unknown, one can estimate Σ consistently by $\hat{\Sigma}$, where

$$\hat{\Sigma} = \begin{bmatrix} \hat{\pi}_1^2\hat{\theta}_1 & \hat{\pi}_1\hat{\pi}_2\hat{\theta}_1 & \hat{\pi}_1\hat{\pi}_3\hat{\theta}_1 & \hat{\pi}_1\hat{\pi}_4\hat{\theta}_1 & \cdot\ \cdot\ \cdot & \hat{\pi}_1\hat{\pi}_{(m-1)}\hat{\theta}_1 & \hat{\pi}_1\hat{\pi}_m\hat{\theta}_1 \\ & \hat{\pi}_2^2\hat{\theta}_2 & \hat{\pi}_2\hat{\pi}_3\hat{\theta}_2 & \hat{\pi}_2\hat{\pi}_4\hat{\theta}_2 & \cdot\ \cdot\ \cdot & \hat{\pi}_2\hat{\pi}_{(m-1)}\hat{\theta}_2 & \hat{\pi}_2\hat{\pi}_m\hat{\theta}_2 \\ & & \hat{\pi}_3^2\hat{\theta}_3 & \hat{\pi}_3\bar{\pi}_4\hat{\theta}_3 & \cdot\ \cdot\ \cdot & \hat{\pi}_3\hat{\pi}_{(m-1)}\hat{\theta}_3 & \hat{\pi}_3\hat{\pi}_m\hat{\theta}_3 \\ & & & \hat{\pi}_4^2\hat{\theta}_4 & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \hat{\pi}_{(m-1)}^2\hat{\theta}_{(m-1)} & \hat{\pi}_{(m-1)}\hat{\pi}_m\hat{\theta}_{(m-1)} \\ & & & & & & \hat{\pi}_m^2\hat{\theta}_m \end{bmatrix},$$

$$\hat{\theta}_j = \frac{n - x_j^*}{nx_j^*},$$

and

$$\hat{\pi}_j = \frac{1}{\ell n \dfrac{x_j^*}{n}}.$$

The modified estimators of $\hat{\alpha}$ and $\hat{\beta}$ of α and β are given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (S'\hat{\Sigma}^{-1}S)^{-1}S'\hat{\Sigma}^{-1}Y$$

$$= \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where

$$a = \sum_{i=1}^{m} \hat{\sigma}_{ii} + 2\sum_{i=2}^{m} \hat{\sigma}_{i-1,i},$$

$$b = \sum_{i=1}^{m} t_i\hat{\sigma}_{ii} + \sum_{i=2}^{m} (t_{i-1} + t_i)\hat{\sigma}_{i-1,i},$$

$$c = \sum_{i=1}^{m} t_i^2\hat{\sigma}_{ii} + 2\sum_{i=2}^{m} t_{i-1}t_i\hat{\sigma}_{i-1,i},$$

$$d_1 = \sum_{i=1}^{m} y_i\hat{\sigma}_{ii} + \sum_{i=2}^{m} (y_{i-1} + y_i)\hat{\sigma}_{i-1,i},$$

$$d_2 = \sum_{i=1}^{m} t_i y_i\hat{\sigma}_{ii} + \sum_{i=2}^{m} (y_i t_{i-1} + t_i y_{i-1})\hat{\sigma}_{i-1,i}.$$

Therefore,

$$\hat{\alpha} = \frac{cd_1 - bd_2}{ac - b^2},$$

and

$$\hat{\beta} = \frac{ad_2 - bd_1}{ac - b^2}.$$

As an illustration, we fit the Weibull distribution to the Part Cracking Data given in Table 8.1. Table 9.1 presents the observed cracked parts and the expected cracked parts under the maximum likelihood and linear model methods. Under the linear model method, the distribution of the lifetime T is Weibull (0.0081, 1.1037); and under the maximum likelihood method, the distribution of T is Weibull (0.00178, 1.4825).

<u>Table 9.1</u>

Fitting Weibull Distribution to the Part Cracking Data

Inspection Intervals	Observed Number cracked	Expected Number cracked (Linear model)	Expected Number cracked (Maximum Likelihood)
(0.00, 6.12]	5	9.71	4.30
(6.12, 19.92]	16	23.32	18.96
(19.92, 29.64]	12	15.31	16.30
(29.64, 35.40]	18	8.46	9.96
(35.40, 39.72]	18	6.04	7.44
(39.72, 52.32]	8	16.09	20.88
(52.32, 63.48]	17	12.43	16.77
(63.48, ∞)	73	75.64	72.40
Total	167	167	167

The chi-square goodness of fit values are

$\chi^2_{lr} = 45.578\ (p < 0.001)$,

and

$\chi^2_{ml} = 31.142\ (p < 0.001)$.

The chi-squared values are significant. The null hypothesis that the underlying distribution is Weibull is rejected.

9.3 Linear Model Approach for the Lognormal Distribution

Note that the random variables

$$Y_j = \Phi^{-1}\left(1 - \sum_{i=j+1}^{m+1} \frac{X_i}{n}\right), \quad j = 1, 2, \ldots, m.$$

have expectations,

$$E(Y_j) = E\Phi^{-1}\left(1 - \frac{X_j^*}{n}\right)$$
$$\cong -\frac{\mu}{\sigma} + \frac{1}{\sigma}\ell n t_j,$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = -\mu/\sigma$, $\beta = 1/\sigma$, and $s_j = \ell n\, t_j$, $j = 1, 2, \ldots, m$. Using the Δ-method again, we get

$$\begin{aligned} \mathrm{Var}(Y_j) &= \mathrm{Var}(\Phi^{-1}(1-\frac{X_j^*}{n})) \\ &\cong \mathrm{Var}(X_j^*)\left[\frac{d}{dX_j^*}\Phi^{-1}(1-\frac{X_j^*}{n})\right]^2_{X_j^*=EX_j^*}. \end{aligned}$$

Thus,

$$\mathrm{Var}(Y_j) \cong \frac{2\pi}{n} q_j(1-q_j)\, e^{\left[\Phi^{-1}(1-q_j)\right]^2},$$

and for $j < k$,

$$\begin{aligned} \mathrm{Cov}(Y_j, Y_k) &= \mathrm{Cov}\left[\Phi^{-1}(1-\frac{X_j^*}{n}),\ \Phi^{-1}(1-\frac{X_k^*}{n})\right] \\ &\cong \mathrm{Cov}(X_j^*, X_k^*)\left[\frac{d}{dX_j^*}\Phi^{-1}(1-\frac{X_j^*}{n})\right]\left[\frac{d}{dX_k^*}\Phi^{-1}(1-\frac{X_k^*}{n})\right]_{\substack{X_k^*=EX_k^* \\ X_j^*=EX_j^*}} \\ &= \frac{2\pi}{n} q_k(1-q_j)\, e^{\frac{\left[\Phi^{-1}(1-q_j)\right]^2}{2}}\, e^{\frac{\left[\Phi^{-1}(1-q_k)\right]^2}{2}}. \end{aligned}$$

In general, for $j \neq k$,

$$\mathrm{Cov}(Y_j,\ Y_k) = \frac{2\pi}{n} q_s(1-q_r)\, e^{-\frac{\left[\Phi^{-1}(1-q_j)\right]^2}{2}}\, e^{-\frac{\left[\Phi^{-1}(1-q_k)\right]^2}{2}},$$

where $r = \mathrm{Min}\ (j, k)$ and $s = \mathrm{Max}\ (j, k)$.

Let us introduce a special notation exploiting the special structure of these variances and covariances. Let

$$\theta_j = \frac{2\pi}{n}(1 - q_j)e^{-\frac{\left[\Phi^{-1}(1-q_j)\right]^2}{2}},$$

and

$$\pi_k = q_k e^{-\frac{\left[\Phi^{-1}(1-q_k)\right]^2}{2}}.$$

Then,

$$\mathrm{Var}(Y_j) = \theta_j \pi_j, \; j = 1, 2, \ldots, m,$$

and for $j \neq k$,

$$\mathrm{Cov}(Y_j, Y_k) = \theta_r \pi_s,$$

where $r = \mathrm{Min}\,(j, k)$ and $s = \mathrm{Max}\,(j, k)$.

These computations can be summarized as follows:

$$E(Y) = E\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \cdot \\ \cdot \\ Y_m \end{pmatrix} = \begin{pmatrix} 1 & \ell nt_1 \\ 1 & \ell nt_2 \\ 1 & \ell nt_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \ell nt_m \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S\beta,$$

the dispersion matrix Σ of Y is given by

$$\Sigma = \begin{bmatrix} \pi_1\theta_1 & \pi_2\theta_1 & \pi_3\theta_1 & \pi_4\theta_1 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_1 & \pi_m\theta_1 \\ & \pi_2\theta_2 & \pi_3\theta_2 & \pi_4\theta_2 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_2 & \pi_m\theta_2 \\ & & \pi_3\theta_3 & \pi_4\theta_3 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_3 & \pi_m\theta_3 \\ & & & \pi_4\theta_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \pi_{(m-1)}\theta_{(m-1)} & \pi_m\theta_{(m-1)} \\ & & & & & & & & \pi_m\theta_m \end{bmatrix}.$$

Using the data one can estimate the dispersion matrix,

$$\Sigma = \begin{bmatrix} \hat{\pi}_1\hat{\theta}_1 & \hat{\pi}_2\hat{\theta}_1 & \hat{\pi}_3\hat{\theta}_1 & \hat{\pi}_4\hat{\theta}_1 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_1 & \hat{\pi}_m\hat{\theta}_1 \\ & \hat{\pi}_2\hat{\theta}_2 & \hat{\pi}_3\hat{\theta}_2 & \hat{\pi}_4\hat{\theta}_2 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_2 & \hat{\pi}_m\hat{\theta}_2 \\ & & \hat{\pi}_3\hat{\theta}_3 & \hat{\pi}_4\hat{\theta}_3 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_3 & \hat{\pi}_m\hat{\theta}_3 \\ & & & \hat{\pi}_4\hat{\theta}_4 & \cdots & \cdot & \cdot \\ & & & & \cdots & \cdot & \cdot \\ & & & & \cdot\ \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \hat{\pi}_{(m-1)}\hat{\theta}_{(m-1)} & \hat{\pi}_m\hat{\theta}_{(m-1)} \\ & & & & & & \hat{\pi}_m\hat{\theta}_m \end{bmatrix},$$

where

$$\hat{\theta}_j = \frac{2\pi}{n}\left(1-\frac{x_j^*}{n}\right)e^{-\frac{\left[\Phi^{-1}\left(1-\frac{x_j^*}{n}\right)\right]^2}{2}},$$

and

$$\pi_k = \frac{x_k^*}{n}e^{-\frac{\left[\Phi^{-1}\left(1-\frac{x_k^*}{n}\right)\right]^2}{2}}.$$

If Σ is known, the Best Linear Unbiased Estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β are given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = (S'\Sigma^{-1}S)^{-1}S\Sigma^{-1}Y.$$

Since Σ is unknown, the estimate $\hat{\Sigma}$ of Σ is used, i.e.,

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = (S\hat{\Sigma}^{-1}S)^{-1}S\hat{\Sigma}^{-1}Y.$$

94 Linear Model Approach for the Pareto Distribution

Note that the random variables

$$Y_j = \ell n\left(\sum_{i=j+1}^{m+1} \frac{X_i}{n}\right), \quad j = 1, 2, .., m.$$

have expectations,

$$E(Y_j) \cong \ell n\left(\frac{a}{t_j}\right),$$

which can be written as

$$E(Y_j) \cong \beta s_j,$$

where $\beta = \gamma$, $s_j = \ell n(\frac{a}{t_j})$, $j = 1, 2, ..., m$. Using the Δ-method again, we get

$$\begin{aligned} Var(Y_j) &= Var\left(\ell n\left(\frac{X_j^*}{n}\right)\right) \\ &\cong Var(X_j^*)\left[\frac{d}{dX_j^*}\ell n\left(\frac{X_j^*}{n}\right)\right]^2_{X_j^*=EX_j^*} \\ &= \frac{1-q_j}{nq_j}, \end{aligned}$$

and for $j < k$,

$$Cov(Y_j, Y_k) = Cov\left(\ell n\left(\frac{X_j^*}{n}\right), \ell n\left(\frac{X_k^*}{n}\right)\right)$$

$$= \frac{1-q_j}{nq_j}.$$

In general, for $j \neq k$,

$$Cov(Y_j,\ Y_k) = \frac{1-q_s}{nq_s},$$

where $s = Min(j, k)$.

The variances and covariances of Y_i's have the following properties:

$$Var(Y_1) = Cov(Y_1, Y_2) = Cov(Y_1, Y_3) = ... = Cov(Y_1, Y_m);$$

$$Var(Y_2) = Cov(Y_2, Y_3) = Cov(Y_2, Y_4) = ... = Cov(Y_2, Y_m);$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$Var(Y_{m-1}) = Cov(Y_{m-1}, Y_m).$$

Let us introduce a special notation exploiting the special structure of these variances and covariances. Let

$$\theta_k = \mathrm{Var}(Y_k) = \frac{p_k}{nq_k}, \; k = 1\ , 2, .., m.$$

These computations can be summarized as follows:

$$E(Y) = E\left(\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \cdot \\ \cdot \\ Y_m \end{pmatrix}\right) =. \begin{pmatrix} \frac{a}{t_1} \\ \frac{a}{t_2} \\ \frac{a}{t_3} \\ \cdot \\ \cdot \\ \frac{a}{t_m} \end{pmatrix} = S\beta,$$

the dispersion matrix Σ of Y is given by

$$\Sigma = \begin{pmatrix} \theta_1 & \theta_1 & \theta_1 & \theta_1 & \theta_1 & . & . & . & \theta_1 & \theta_1 \\ & \theta_2 & \theta_2 & \theta_2 & \theta_2 & . & . & . & \theta_2 & \theta_2 \\ & & \theta_3 & \theta_3 & \theta_3 & . & . & . & \theta_3 & \theta_3 \\ & & & \theta_4 & \theta_4 & . & . & . & \theta_4 & \theta_4 \\ & & & & \theta_5 & . & . & . & \theta_5 & \theta_5 \\ & & & & & . & . & . & . & . \\ & & & & & & . & . & . & . \\ & & & & & & & . & . & . \\ & & & & & & & & \theta_{(m-1)} & \theta_{(m-1)} \\ & & & & & & & & & \theta_m \end{pmatrix}.$$

Using the data, one can estimate the dispersion matrix,

$$\hat{\Sigma} = \begin{pmatrix} \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & . & . & . & \hat{\theta}_1 & \hat{\theta}_1 \\ & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & . & . & . & \hat{\theta}_2 & \hat{\theta}_2 \\ & & \hat{\theta}_3 & \hat{\theta}_3 & \hat{\theta}_3 & . & . & . & \hat{\theta}_3 & \hat{\theta}_3 \\ & & & \hat{\theta}_4 & \hat{\theta}_4 & . & . & . & \hat{\theta}_4 & \hat{\theta}_4 \\ & & & & \hat{\theta}_5 & . & . & . & \hat{\theta}_5 & \hat{\theta}_5 \\ & & & & & . & . & . & . & . \\ & & & & & & . & . & . & . \\ & & & & & & & . & . & . \\ & & & & & & & & \hat{\theta}_{(m-1)} & \hat{\theta}_{(m-1)} \\ & & & & & & & & & \hat{\theta}_m \end{pmatrix},$$

where

$$\hat{\theta}_j = \frac{n - x_j^*}{n x_j^*}.$$

Let $\Sigma^{-1} = (\sigma_{ij})$. The inverse Σ^{-1} is a tri-diagonal matrix. If Σ is known, the best linear unbiased estimator $\hat{\beta}$ of β is given by

$$\hat{\beta} = (S'\Sigma^{-1}S)^{-1}S'\Sigma^{-1}Y$$

$$= \frac{\sum_{i=1}^{m} \frac{y_i \sigma_{ii}}{t_i} + \sum_{i=2}^{m} \left(\frac{y_i}{t_{i-1}} + \frac{y_{i-1}}{t_i}\right)\sigma_{i-1,i}}{a\left[\sum_{i=1}^{m} \frac{\sigma_{ii}}{t_i^2} + 2\sum_{i=2}^{m} \frac{\sigma_{i-1,i}}{t_{i-1}t_i}\right]}.$$

Since Σ is unknown, one can estimate Σ consistently by $\hat{\Sigma}$, where

$$\hat{\Sigma} = \begin{pmatrix} \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \hat{\theta}_1 & \cdot & \cdot & \cdot & \hat{\theta}_1 & \hat{\theta}_1 \\ & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & \hat{\theta}_2 & \cdot & \cdot & \cdot & \hat{\theta}_2 & \hat{\theta}_2 \\ & & \hat{\theta}_3 & \hat{\theta}_3 & \hat{\theta}_3 & \cdot & \cdot & \cdot & \hat{\theta}_3 & \hat{\theta}_3 \\ & & & \hat{\theta}_4 & \hat{\theta}_4 & \cdot & \cdot & \cdot & \hat{\theta}_4 & \hat{\theta}_4 \\ & & & & \hat{\theta}_5 & \cdot & \cdot & \cdot & \hat{\theta}_5 & \hat{\theta}_5 \\ & & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & & \hat{\theta}_{(m-1)} & \hat{\theta}_{(m-1)} \\ & & & & & & & & & \hat{\theta}_m \end{pmatrix},$$

and

$$\hat{\theta}_j = \frac{n - x_j^*}{n x_j^*}.$$

The modified estimator $\hat{\beta}$ of β is given by

$$\hat{\beta} = (S'\Sigma^{-1}S)^{-1}S'\Sigma^{-1}Y$$

$$= \frac{\sum_{i=1}^{m} \frac{y_i \hat{\sigma}_{ii}}{t_i} + \sum_{i=2}^{m} \left(\frac{y_i}{t_{i-1}} + \frac{y_{i-1}}{t_i}\right)\hat{\sigma}_{i-1,i}}{a\left[\sum_{i=1}^{m} \frac{\hat{\sigma}_{ii}}{t_i^2} + 2\sum_{i=2}^{m} \frac{\hat{\sigma}_{i-1,i}}{t_{i-1}t_i}\right]}.$$

where

$$\hat{\Sigma}^{-1} = (\hat{\sigma}_{ij}).$$

9.5 Linear Model Approach for the Logistic Distribution

Note that the random variables

$$Y_j = \ell n\left(\frac{1}{\sum_{i=j+1}^{m+1} \frac{X_i}{n}} - 1\right), \; j = 1,\ 2,\ \ldots,\ m.$$

have expectations,

$$E(Y_j) = E\ell n(\frac{n}{X_j^*} - 1)$$
$$\cong n\lambda + \gamma \ell n t_j,$$

which can be written as

$$E(Y_j) \cong \alpha + \beta s_j,$$

where $\alpha = \ell n\, \lambda$, and $\beta = \gamma$, and $s_j = \ell n\, t_j$, j = 1, 2, ...,m. Using the Δ-method again, we get

$$\mathrm{Var}(Y_j) = \mathrm{Var}(\ell n(\frac{n}{X_j^*} - 1))$$
$$\cong \mathrm{Var}(X_j^*)\left[\frac{d}{dX_j^*}\ell n(\frac{n}{X_j^*} - 1)\right]^2_{X_j^* = EX_j^*}$$
$$= \frac{1}{nq_j(1-q_j)},$$

and for j < k,

$$\mathrm{Cov}(Y_j, Y_k) = \mathrm{Cov}(\ell n(\frac{n}{X_j^*} - 1), \ell n(\frac{n}{X_k^*} - 1)))$$
$$= \mathrm{Cov}(X_j^*, X_k^*)\left[\frac{d}{dX_j^*}\ell n(\frac{n}{X_j^*} - 1)\right]\left[\frac{d}{dX_k^*}\ell n(\frac{n}{X_k^*} - 1)\right]_{\substack{X_k^* = EX_k^* \\ X_j^* = EX_j^*}}$$
$$= \mathrm{Cov}(\sum_{i=j+1}^{m+1} X_j, \sum_{i=k+1}^{m+1} X_k)\left[\frac{-n}{X_j^*(n - X_j^*)}\right]\left[\frac{-n}{X_k^*(n - X_k^*)}\right]_{\substack{X_k^* = EX_k^* \\ X_j^* = EX_j^*}}$$
$$= \left[nq_k(1-q_k) + \sum_{r=j+1}^{k}\sum_{s=k+1}^{m+1}(-np_r p_s)\right]\left[\frac{-n}{nq_j(n - nq_j)}\right]\left[\frac{-n}{nq_k(n - nq_k)}\right]$$
$$= \frac{1}{nq_j(1-q_k)}.$$

In general, for $i \neq k$

$$\mathrm{Cov}(Y_j, Y_k) = \frac{1}{nq_r(1-q_s)},$$

where r = Min (j, k) and s = Max (j, K).

Let us introduce a special notation exploiting the special structure of these variances and covariances. Let

$$\theta_j = \frac{1}{nq_j},$$

and

$$\pi_k = \frac{1}{1-q_k}.$$

Then,

$$\mathrm{Var}(Y_j) = \theta_j \pi_j, \quad j = 1, 2, \ldots, m,$$

and for $j \neq k$,

$$\mathrm{Cov}(Y_j, Y_k) = \theta_r \pi_s,$$

where $r = \mathrm{Min}\,(j, k)$ and $s = \mathrm{Max}\,(j, k)$.

These computations can be summarized as follows:

$$E(Y) = E\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \cdot \\ \cdot \\ Y_m \end{pmatrix} = \begin{pmatrix} 1 & \ell nt_1 \\ 1 & \ell nt_2 \\ 1 & \ell nt_3 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \ell nt_m \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = S\beta,$$

the dispersion matrix Σ of Y is given by

$$\Sigma = \begin{bmatrix} \pi_1\theta_1 & \pi_2\theta_1 & \pi_3\theta_1 & \pi_4\theta_1 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_1 & \pi_m\theta_1 \\ & \pi_2\theta_2 & \pi_3\theta_2 & \pi_4\theta_2 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_2 & \pi_m\theta_2 \\ & & \pi_3\theta_3 & \pi_4\theta_3 & \cdot & \cdot & \cdot & \pi_{(m-1)}\theta_3 & \pi_m\theta_3 \\ & & & \pi_4\theta_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot & \cdot \\ & & & & & & & \pi_{(m-1)}\theta_{(m-1)} & \pi_m\theta_{(m-1)} \\ & & & & & & & & \pi_m\theta_m \end{bmatrix}.$$

Using the data one can estimate the dispersion matrix,

$$
\Sigma = \begin{bmatrix}
\hat{\pi}_1\hat{\theta}_1 & \hat{\pi}_2\hat{\theta}_1 & \hat{\pi}_3\hat{\theta}_1 & \hat{\pi}_4\hat{\theta}_1 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_1 & \hat{\pi}_m\hat{\theta}_1 \\
 & \hat{\pi}_2\hat{\theta}_2 & \hat{\pi}_3\hat{\theta}_2 & \hat{\pi}_4\hat{\theta}_2 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_2 & \hat{\pi}_m\hat{\theta}_2 \\
 & & \hat{\pi}_3\hat{\theta}_3 & \hat{\pi}_4\hat{\theta}_3 & \cdots & \hat{\pi}_{(m-1)}\hat{\theta}_3 & \hat{\pi}_m\hat{\theta}_3 \\
 & & & \hat{\pi}_4\hat{\theta}_4 & \cdots & \cdot & \cdot \\
 & & & & \cdots & \cdot & \cdot \\
 & & & & \cdot\ \cdot & \cdot & \cdot \\
 & & & & \cdot & \cdot & \cdot \\
 & & & & & \hat{\pi}_{(m-1)}\hat{\theta}_{(m-1)} & \hat{\pi}_m\hat{\theta}_{(m-1)} \\
 & & & & & & \hat{\pi}_m\hat{\theta}_m
\end{bmatrix},
$$

where

$$\hat{\theta}_j = \frac{1}{x_j^*},$$

and

$$\pi_k = \frac{n}{n - x_k^*}.$$

If Σ is known, the Best Linear Unbiased Estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β are given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (S'\Sigma^{-1}S)^{-1}S'\Sigma^{-1}Y.$$

Since Σ is unknown, the estimate $\hat{\Sigma}$ of Σ is used, i.e.,

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = (S'\hat{\Sigma}^{-1}S')^{-1}S'\hat{\Sigma}^{-1}Y.$$

9.6 Linear Model Approach for the Extreme Value Distribution

The probability density function of Extreme value distribution is given by

$$f_X(x) = \gamma\, e^{\gamma(x-\theta)}\, e^{-e^{\gamma(x-\theta)}}, \qquad -\infty < x < \infty.$$

where $-\infty < \theta < \infty$ and $\gamma > 0$ are the unknown parameters of the model. Let

$$T = e^X.$$

It can be shown that the probability density function of T is given by

$$f_T(t) = \lambda\gamma\, t^{\gamma-1}\, e^{-\lambda t^{\gamma-1}}, \quad 0 < t < \infty.$$

where $\lambda = e^{-\gamma\theta}$, which is the probability density function of the Weibull distribution with parameters γ and λ. Therefore, to find the linear model estimators of the Extreme value distribution parameters, we fit the Weibull distribution to the $T = e^X$ variable. The estimate $\hat{\gamma}$ of γ is equal to $\hat{\gamma}$, and the estimate $\hat{\theta}$ of θ is equal to $-\ell n\hat{\lambda} / \hat{\gamma}$.

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